

On the Gauge Aspects of Gravity[†]

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ABSTRACT

We give a short outline, in Sec. 2, of the historical development of the gauge idea as applied to internal ($U(1)$, $SU(2)$, ...) and external (R^4 , $SO(1,3)$, ...) symmetries and stress the fundamental importance of the corresponding conserved currents. In Sec. 3, experimental results with neutron interferometers in the gravitational field of the earth, as interpreted by means of the equivalence principle, can be predicted by means of the Dirac equation in an accelerated and rotating reference frame. Using the Dirac equation in such a non-inertial frame, we describe how in a gauge-theoretical approach (see Table 1) the Einstein-Cartan theory, residing in a Riemann-Cartan spacetime encompassing torsion and curvature, arises as the simplest gravitational theory. This is set in contrast to the Einsteinian approach yielding general relativity in a Riemannian spacetime. In Secs. 4 and 5 we consider the conserved energy-momentum current of matter and gauge the associated translation subgroup. The Einsteinian teleparallelism theory which emerges is shown to be equivalent, for spinless matter and for electromagnetism, to general relativity. Having successfully gauged the translations, it is straightforward to gauge the four-dimensional affine group $R^4 \rtimes GL(4, R)$ or its Poincaré subgroup $R^4 \rtimes SO(1, 3)$. We briefly report on these results in Sec. 6 (metric-affine geometry) and in Sec. 7 (metric-affine field equations (111, 112, 113)). Finally, in Sec. 8, we collect some models, currently under discussion, which bring life into the metric-affine gauge framework developed.

[†] Proc. of the 14th Course of the School of Cosmology and Gravitation on *Quantum Gravity*, held at Erice, Italy, May 1995, P.G. Bergmann, V. de Sabbata, and H.-J. Treder, eds. (World Scientific, Singapore 1996) to be published.

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From a letter of A. Einstein to F. Klein of 1917 March 4 (translation)⁷⁰:

“... Newton’s theory ... represents the gravitational field in a seemingly complete way by means of the potential Φ . This description proves to be wanting; the functions $g_{\mu\nu}$ take its place. But I do not doubt that the day will come when that description, too, will have to yield to another one, for reasons which at present we do not yet surmise. I believe that this process of deepening the theory has no limits ... ”

1. Introduction

- What can we learn if we look at gravity and, more specifically, at general relativity theory (GR) from the point of view of classical gauge field theory? This is the question underlying our present considerations. The answer
- leads to a better understanding of the interrelationship between the *metric* and *affine* properties of spacetime and of the *group* structure related to gravity. Furthermore, it
- suggests certain classical field-theoretical generalizations of Einstein’s theory, such as Einstein–Cartan theory, Einsteinian teleparallelism theory, Poincaré gauge theory, Metric-Affine Gravity, that is, it leads to a deepening of the insight won by GR.

We recently published a fairly technical review article on our results²⁹. These lectures can be regarded as a down-to-earth introduction into that subject. We refrain from citing too many articles since we gave an overview^a of the existing literature in ref.⁽²⁹⁾.

2. Remarks on the history of the gauge idea

2.1. General relativity and Weyl’s $U(1)$ -gauge theory

Soon after Einstein in 1915/16 had proposed his gravitational theory, namely general relativity (GR), Weyl extended it in 1918 in order to include – besides gravitation – electromagnetism in a unified way. Weyl’s theoretical concept was that of recalibration or *gauge invariance* of length. In Weyl’s opinion, the integrability of length in GR is a remnant of an era dominated by action-at-a-distance theories which should be abandoned. In other words, if in GR we displace a meter stick from one point of spacetime to another one, it keeps its length, i.e., it can be used as a standard

^aIn the meantime we became aware that in our Physics Reports we should have cited additionally the work of Mistura⁵⁸ on the physical interpretation of torsion, of Pascual-Sánchez^{71,72,73} on teleparallelism, inter alia, of Perlick⁷⁶ on observers in Weyl spacetimes, and that of Ponomariov and Obukhov⁷⁸ on metric-affine spacetimes and gravitational theories with quadratic Lagrangians.

of length throughout spacetime; an analogous argument is valid for a clock. In contrast, Weyl's unified theory of gravitation and electromagnetism of 1918 is set up in such a way that the unified Lagrangian is invariant under recalibration or re-gauging.

For that purpose, Weyl extended the geometry of spacetime from the (pseudo-) Riemannian geometry with its Levi-Civita connection $\Gamma_{\alpha\beta}^{\{\}}$ to a Weyl space with an additional (Weyl) covector field $Q = Q_\alpha \vartheta^\alpha$, where ϑ^α denotes the field of coframes of the underlying four-dimensional differentiable manifold. The Weyl connection one-form reads

$$\Gamma_{\alpha\beta}^{\text{W}} = \Gamma_{\alpha\beta}^{\{\}} + \frac{1}{2}(g_{\alpha\beta} Q - \vartheta_\alpha Q_\beta + \vartheta_\beta Q_\alpha). \quad (1)$$

The additional freedom of having a new one-form (or covector) field Q at one's disposal was used by Weyl in order to accommodate Maxwell's field. He identified the electromagnetic potential A with Q .

Weyl's theory turned out to be non-viable, at least in the sense and on the level of ordinary length and time measurement. However, his concept of gauge invariance survived in the following way: When quantum mechanics was developed, it became clear that (in the Schrödinger representation) the wave function Ψ of an electron, for example, is only determined up to an arbitrary phase ϕ :

$$\Psi \longrightarrow e^{i\phi}\Psi, \quad \phi = \text{const}. \quad (2)$$

The set of all phase transformations builds up the one-dimensional Abelian Lie group $U(1)$ of unitary transformations. If one substitutes, according to (2), the wave function in the Dirac equation by a phase transformed wave function, no observables will change; they are invariant under 'rigid', i.e., constant phase transformations. This is an elementary fact of quantum mechanics.

In 1929, Weyl revitalized his gauge idea: Isn't it against the spirit of field theory to implement a rigid phase transformation (2) 'at once' all over spacetime, he asked. Shouldn't we postulate a $U(1)$ invariance under a spacetime dependent change of the phase instead:

$$\Psi \longrightarrow e^{i\phi(x)}\Psi, \quad \phi = \phi(x)? \quad (3)$$

If one does it, the original invariance of the observables is lost under the new 'soft' transformations. In order to kill the invariance violating terms, one has to introduce a compensating potential one-form A with values in the Lie algebra of $U(1)$, which transforms under the soft transformations in a suitable form. This couples A in a well determined way to the wave function of the electron, and, if one insists that the $U(1)$ -potential A has its own physical degrees of freedom, then the field strength $F := dA$ is non-vanishing and the coupled Dirac Lagrangian has to be amended with a kinetic term quadratic in F . In this way one can reconstruct the whole classical Dirac-Maxwell theory from the naked Dirac equation together with the postulate of soft phase invariance. Because of Weyl's original terminology, one still talks about $U(1)$ -gauge invariance – and not about $U(1)$ -phase invariance, what it really is. Thus

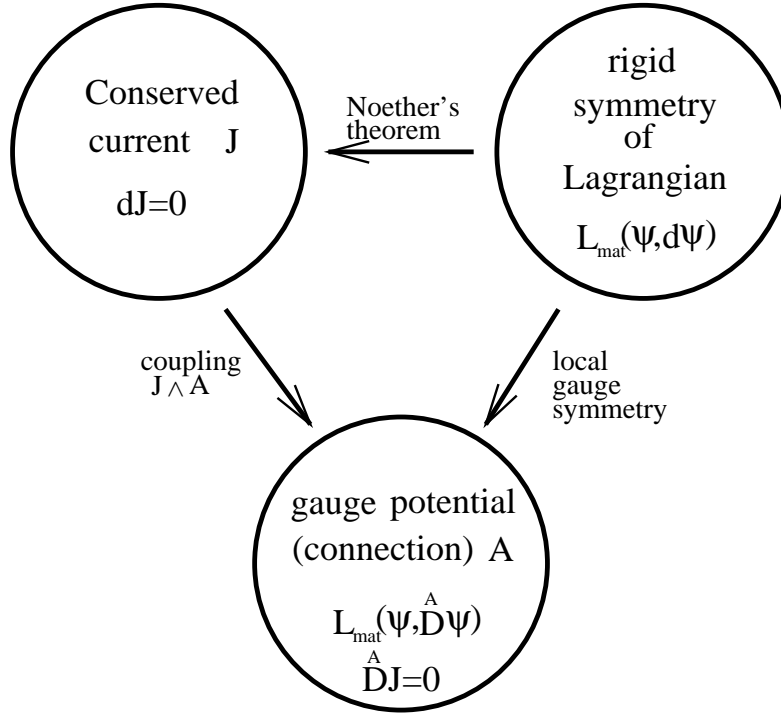


Fig. 1. The structure of a gauge theory à la Yang-Mills is depicted in this diagram, which is adapted from Mills⁵³. Let us quote some of his statements on gauge theories: ‘The gauge principle, which might also be described as a principle of *local symmetry*, is a statement about the invariance properties of physical laws. It requires that *every continuous symmetry be a local symmetry...*’ ‘The idea at the core of gauge theory...is the local symmetry principle: *Every continuous symmetry of nature is a local symmetry.*’ The history of gauge theory has been traced back to its beginnings by O’Raifeartaigh⁶⁹, who also gave a compact review of its formalism⁶⁸.

the electromagnetic potential is an appendage to the Dirac field and not related to length recalibration as Weyl originally thought.

2.2. Yang-Mills and the structure of a gauge theory

Yang and Mills, in 1954, generalized the Abelian $U(1)$ -gauge invariance to non-Abelian $SU(2)$ -gauge invariance, taking the (approximately) *conserved isotopic spin current* as their starting point, and, in 1956, Utiyama set up a formalism for the gauging of any semi-simple Lie group, including the Lorentz group $SO(1,3)$. The latter group he considered as essential in GR. We will come back to this topic below.

In any case, the gauge principle historically originated from GR as a concept for removing as many action-at-a-distance concept as possible – as long as the group under consideration is linked to a conserved current. This existence of a conserved current of some matter field Ψ is absolutely vital for the setting-up of a gauge theory. In Fig.1 we sketched the structure underlying a gauge theory: A rigid symmetry of

a Lagrangian induces, via Noether's theorem, a conserved current J , $dJ = 0$. It can happen, however, as it did in the electromagnetic and the $SU(2)$ -case, that a conserved current is discovered first and then the symmetry deduced by a kind of a reciprocal Noether theorem (which is not strictly valid). Generalizing from the gauge approach to the Dirac-Maxwell theory, we continue with the following gauge procedure:

Extending the rigid symmetry to a soft symmetry amounts to turn the constant group parameters ε of the symmetry transformation on the fields Ψ to functions of spacetime, $\varepsilon \rightarrow \varepsilon(x)$. This affects the transformation behavior of the matter Lagrangian which usually contains derivatives $d\Psi$ of the field Ψ : The soft symmetry transformations on $d\Psi$ generate terms containing derivatives $d\varepsilon(x)$ of the spacetime-dependent group parameters which spoil the former rigid invariance. In order to counterbalance these terms, one is forced to introduce a compensating field $A = A_i^a \tau_a dx^i$ (a =Lie-algebra index, τ_a =generators of the symmetry group) – nowadays called *gauge potential* – into the theory. The one-form A turns out to have the mathematical meaning of a Lie-algebra valued *connection*. It acts on the components of the fields Ψ with respect to some reference frame, indicating that it can be properly represented as the connection of a frame bundle which is associated to the symmetry group. Thereby it is possible to replace in the matter Lagrangian the exterior derivative of the matter field by a gauge-covariant exterior derivative,

$$d \longrightarrow \overset{A}{D} := d + A, \quad L_{\text{mat}}(\Psi, d\Psi) \longrightarrow L_{\text{mat}}(\Psi, \overset{A}{D}\Psi). \quad (4)$$

This is called *minimal coupling* of the matter field to the new gauge interaction. The connection A is made to a true dynamical variable by adding a corresponding kinematic term V to the minimally coupled matter Lagrangian. This supplementary term has to be gauge invariant such that the gauge invariance of the action is kept.

Gauge invariance of V is obtained by constructing it from the field strength $F = \overset{A}{D}A$, $V = V(F)$. Hence the gauge Lagrangian V , as in Maxwell's theory, is assumed to depend only on $F = dA$, not, however, on its derivatives dF, d^*dF, \dots . Therefore the field equation will be of second order in the gauge potential A . In order to make it quasilinear, that is, linear in the second derivatives of A , the gauge Lagrangian must depend on F no more than quadratically. Accordingly, with the general ansatz $V = F \wedge H$, where the field momentum or "excitation" H is implicitly defined by $H = -\partial V / \partial F$, the H has to be linear in F under those circumstances.

By construction, the gauge potential in the Lagrangians couples to the conserved current one started with – and the original conservation law, in case of a non-Abelian symmetry, gets modified and is only gauge covariantly conserved,

$$dJ = 0 \quad \longrightarrow \quad \overset{A}{D}J = 0, \quad J = \partial L_{\text{mat}} / \partial A. \quad (5)$$

The physical reason for this modification is that the gauge potential itself contributes a piece to the current, that is, the gauge field (in the non-Abelian case) is charged.

For instance, the Yang-Mills gauge potential B^a carries isotopic spin, since the $SU(2)$ -group is non-Abelian, whereas the electromagnetic potential, being $U(1)$ -valued and Abelian, is electrically uncharged.

2.3. Gravity and the Utiyama-Sciama-Kibble approach

Let us come back to Utiyama (1956). He gauged the Lorentz group $SO(1,3)$, inter alia. Using some ad hoc assumptions, like the postulate of the symmetry of the connection, he was able to recover GR. This procedure is not completely satisfactory, as is also obvious from the fact that the conserved current, linked to the Lorentz group, is the angular momentum current. And this current alone cannot represent the source of gravity. Accordingly, it was soon pointed out by Sciama and Kibble (1961) that it is really the *Poincaré group* $R^4 \rtimes SO(1,3)$, the semi-direct product of the translation and the Lorentz group, which underlies gravity. They found a slight generalization of GR, the so-called Einstein-Cartan theory (EC), which relates – in a Einsteinian manner – the mass-energy of matter to the curvature and – in a novel way – the material *spin* to the torsion of spacetime. In contrast to the Weyl connection (1), the spacetime in EC is still metric compatible, i.e. governed by a Riemann-Cartan^b (RC) geometry. Torsion is admitted according to

$$\Gamma_{\alpha\beta}^{\text{RC}} = \Gamma_{\alpha\beta}^{\{\}} - \frac{1}{2} [e_\alpha \rfloor T_\beta - e_\beta \rfloor T_\alpha - (e_\alpha \rfloor e_\beta \rfloor T_\gamma) \vartheta^\gamma] . \quad (6)$$

Incidentally, approaches to the gauging of the Poincaré group on a *fixed* Minkowski background yield effectively similar structures, see Wiesendanger⁹⁶.

In order to fix the notation, let us shortly recapitulate the structures emerging in spacetime geometry. A four-dimensional differential manifold has at each point a tangent space, spanned by the four basis vectors $e_\alpha = e^i_\alpha \partial_i$, with ∂_i as the vectors tangent to the coordinate lines and $\alpha, \beta = 0, 1, 2, 3$, and the cotangent space, spanned by the four one-forms $\vartheta^\beta = e_j^\beta dx^j$, with dx^j as coordinate one-forms. The two types of bases are dual to each other $e_\alpha \rfloor \vartheta^\beta = \delta_\alpha^\beta$, where \rfloor denotes the interior product, i.e., in components, $e^k_\alpha e_k^\beta = \delta_\alpha^\beta$. This is the underlying manifold.

On top of it we specify a linear connection. For Einstein¹⁴ “... the essential achievement of general relativity, namely to overcome ‘rigid’ space (ie the inertial frame), is *only indirectly* connected with the introduction of a Riemannian metric. The directly relevant conceptual element is the ‘displacement field’ (Γ_{ik}^l), which expresses the infinitesimal displacement of vectors. It is this which replaces the parallelism of spatially arbitrarily separated vectors fixed by the inertial frame (ie the equality of corresponding components) by an infinitesimal operation. This makes it possible to construct tensors by differentiation and hence to dispense with the introduction of ‘rigid’ space (the inertial frame). In the face of this, it seems to be of

^bThe terminology is not quite uniform. Borzeskowski and Treder⁹, in their critical evaluation of different gravitational variational principles, call such a geometry a Weyl-Cartan geometry.

secondary importance in some sense that some particular Γ field can be deduced from a Riemannian metric...” In this vein, we introduce a linear connection

$$\Gamma_{\alpha}^{\beta} = \Gamma_{i\alpha}^{\beta} dx^i, \quad (7)$$

with values in the Lie-algebra of the linear group $GL(4, R)$. These 64 components $\Gamma_{i\alpha}^{\beta}(x)$ of the ‘displacement’ field enable us, as pointed out in the quotation by Einstein, to get rid of the rigid spacetime structure of special relativity (SR).

In order to be able to recover SR in some limit, the primary structure of a connection of spacetime has to be enriched by the secondary structure of a metric

$$g = g_{\alpha\beta} \vartheta^{\alpha} \otimes \vartheta^{\beta}, \quad (8)$$

with its 10 component fields $g_{\alpha\beta}(x)$. At least at the present stage of our knowledge, this additional postulate of the existence of a metric seems to lead to the only practicable way to set up a theory of gravity. In some future time one may be able to ‘deduce’ the metric from the connection and some extremal property of the action function – and some people have tried to develop such type of models, but without success so far.

2.4. E. Cartan’s analysis of general relativity and its consequences

Besides the gauge theoretical line of development which, with respect to gravity, culminated in the Sciame-Kibble approach, there was a second line dominated by E. Cartan’s (1923) geometrical analysis of GR. The concept of a linear connection as an independent and primary structure of spacetime, see (7), developed gradually around 1920 from the work of Hessenberg, Levi-Civita, Weyl, Schouten, Eddington, and others. In its full generality it can be found in Cartan’s work. In particular, he introduced the notion of a so-called torsion – in holonomic coordinates this is the antisymmetric and therefore tensorial part of the components of the connection – and discussed Weyl’s unified field theory from a geometrical point of view.

For this purpose, let us tentatively call

$$(g_{\alpha\beta}, \vartheta^{\alpha}, \Gamma_{\alpha}^{\beta}) \quad (9)$$

the *potentials* in a gauge approach to gravity and

$$(Q_{\alpha\beta}, T^{\alpha}, R_{\alpha}^{\beta}) \quad (10)$$

the corresponding *field strengths*. Later, in Sec. 6, inter alia, we will see why this choice of language is appropriate. Here we defined

$$\text{nonmetricity} \quad Q_{\alpha\beta} := -\overset{\Gamma}{D} g_{\alpha\beta}, \quad (11)$$

$$\text{torsion} \quad T^{\alpha} := \overset{\Gamma}{D} \vartheta^{\alpha} = d\vartheta^{\alpha} + \Gamma_{\beta}^{\alpha} \wedge \vartheta^{\beta}, \quad (12)$$

$$\text{curvature} \quad R_{\alpha}^{\beta} := \overset{\Gamma}{D} \Gamma_{\alpha}^{\beta} = d\Gamma_{\alpha}^{\beta} - \Gamma_{\alpha}^{\gamma} \wedge \Gamma_{\gamma}^{\beta}. \quad (13)$$

Then symbolically we have

$$(Q_{\alpha\beta}, T^\alpha, R_\alpha{}^\beta) \sim \overset{\Gamma}{D} (g_{\alpha\beta}, \vartheta^\alpha, \Gamma_\alpha{}^\beta). \quad (14)$$

By means of the field strengths it is straightforward of how to classify the space-time manifolds of the different theories discussed so far:

$$\text{GR (1915)} : \quad Q_{\alpha\beta} = 0, \quad T^\alpha = 0, \quad R_\alpha{}^\beta \neq 0. \quad (15)$$

$$\text{Weyl (1918)} : \quad Q_\gamma{}^\gamma \neq 0, \quad T^\alpha = 0, \quad R_\alpha{}^\beta \neq 0. \quad (16)$$

$$\text{EC (1923/61)} : \quad Q_{\alpha\beta} = 0, \quad T^\alpha \neq 0, \quad R_\alpha{}^\beta \neq 0. \quad (17)$$

Note that Weyl's theory of 1918 requires only a nonvanishing *trace* of the nonmetricity, the *Weyl covector* $Q := Q_\gamma{}^\gamma/4$. For later use we amend this table with the Einsteinian teleparallelism (GR_{||}), which was discussed between Einstein and Cartan in considerable detail (see Debever¹²) and with metric-affine gravity²⁹ (MAG), which presupposes the existence of a connection *and* a (symmetric) metric that are completely independent from each other (as long as the field equations are not solved):

$$\text{GR}_{||} \text{ (1928)} : \quad Q_{\alpha\beta} = 0, \quad T^\alpha \neq 0, \quad R_\alpha{}^\beta = 0. \quad (18)$$

$$\text{MAG (1976)} : \quad Q_{\alpha\beta} \neq 0, \quad T^\alpha \neq 0, \quad R_\alpha{}^\beta \neq 0. \quad (19)$$

Both theories, GR_{||} and MAG, were originally devised as unified field theories with no sources on the right hand sides of their field equations. Today, however, we understand them^{10,29} as gauge type theories with well-defined sources.

Cartan gave a beautiful geometrical interpretation of the notions of torsion and curvature. Consider a vector at some point of a manifold, that is equipped with a connection, and displace it around an infinitesimal (closed) loop by means of the connection such that the (flat) tangent space, where the vector 'lives' in, rolls without gliding around the loop. At the end of the journey²⁹ the loop, mapped into the tangent space, has a small closure failure, i.e. a translational misfit. Moreover, in the case of vanishing nonmetricity $Q_{\alpha\beta} = 0$, the vector underwent a small rotation or – if no metric exists – a small linear transformation. The torsion of the underlying manifold is a measure for the emerging translation and the curvature for the rotation (or linear transformation):

$$\text{translation} \longrightarrow \text{torsion } T^\alpha \quad (20)$$

$$\text{rotation (lin. transf.)} \longrightarrow \text{curvature } R_\alpha{}^\beta. \quad (21)$$

Hence, if your friend tells you that he discovered that torsion is closely related to electromagnetism or to some other nongravitational field – and there are many such 'friends' around, as we can tell you as referees – then you say: 'No, torsion is related to translations, as had been already found by Cartan in 1923.' And translations – we

Fig. 2. The neutron interferometer of the COW-experiment^{11,18}: A neutron beam is split into two beams which travel in different gravitational potentials. Eventually the two beams are reunited and their relative phase shift is measured.

hope that we don't tell you a secret – are, via Noether's theorem, related to energy-momentum^c i.e. to the source of gravity, and to nothing else. We will come back to this discussion in Sec.4.

For the rest of these lectures, unless stated otherwise, we will choose the frame e_α , and hence also the coframe ϑ^β , to be *orthonormal*, that is,

$$g(e_\alpha, e_\beta) \stackrel{*}{=} o_{\alpha\beta} := \text{diag}(- + ++). \quad (22)$$

Then, in a Riemann-Cartan space, we have the convenient antisymmetries

$$\Gamma_{\alpha\beta}^{RC} \stackrel{*}{=} -\Gamma_{\beta\alpha}^{RC} \quad \text{and} \quad R_{\alpha\beta}^{RC} \stackrel{*}{=} -R_{\beta\alpha}^{RC}. \quad (23)$$

3. Einstein's and the gauge approach to gravity

3.1. Neutron matter waves in the gravitational field

^c Not long ago, one of the editors of Physical Review D, Lowell S. Brown, decreed that all papers with (Cartan's) torsion in the title or the abstract are automatically to be rejected. You may guess how the authors of such papers reacted. I was always wondering what this clever physicist would do, if 'energy-momentum' appeared in a title. My guess being that he would not even recognize the need for becoming active again.

Twenty years ago a new epoch began in gravity: **Colella-Overhauser-Werner** measured by interferometric methods a phase shift of the wave function of a neutron caused by the gravitational field of the earth, see Fig.2. The effect could be predicted by studying the Schrödinger equation of the neutron wave function in an external Newtonian potential – and this had been verified by experiment. In this sense nothing really earth-shaking happened. However, for the first time a gravitational effect had been measured the numerical value of which depends on the Planck constant \hbar . Quantum mechanics was indispensable in deriving this phase shift

$$\theta_{\text{grav}} = \frac{m^2 g}{2\pi\hbar^2} \lambda A \sin \alpha \quad (24)$$

(m = mass of the neutron, λ its de Broglie wave length, g = gravitational acceleration, A = area surrounded by the neutron beams, α = angle between the normal vector of the area A and the vector \mathbf{g}).

It was the availability of nearly perfect single silicon crystals of about 10 *cm* length that provided a new tool for X-ray and neutron interferometry. This had first been demonstrated by Bonse and Hart in 1965 for X-rays. After Bonse (1974) and Rauch, Treimer, and Bonse (1974), had shown that this device also works for neutrons, Colella, Overhauser, and Werner (in the following abbreviated by COW) “...used a neutron interferometer to observe the quantum-mechanical phase shift of neutrons caused by their interaction with the Earth’s gravitational field”¹¹. Their experiment is sketched in Fig.3.

They used neutrons cooled to room temperature such that their resulting mean velocity $v_n \simeq 10^{-5} c$ is non-relativistic. Their mass is $m_n = 1.67 \times 10^{-21} kg$, and the de Broglie wave length $\lambda_n := 2\pi\hbar/p \approx 0.2 nm$. A beam of 1 *cm* width enters the first ‘ear’ of the interferometer at a Bragg angle in the range of 20° to 30°. It is *coherently* scattered by planes of atoms perpendicular to the surface of the crystal. This Laue scattering gives rise to a transmitted and a diffracted beam, with opposite Bragg angles. Due to the Borrmann effect, the beam travels through the crystal at first along the planes and the splitting occurs actually only after it emerges from the ear again.

When the interferometer gets rotated in the gravitational field of the earth, the upper and lower beams travel at a vertical distance of about 2 *cm* and encounter a potential difference of $\Delta\varphi/E_{\text{kin}} = (m_n g \Delta h / (1/2) m_n v^2) \approx 10^3 / (3 \times 10^5)^2 \approx 10^{-8}$, which is only a tiny fraction of the kinetic energy. Nevertheless, this leads to a measurable effect on the *phase* of the neutron’s coherent wave which oscillates about $10 cm / \lambda_n \approx 10^9$ times during the horizontal flight. Although the oscillation rate of the upper beam is ‘redshifted’ merely by a factor of 10^{-7} , the upper beam manages to make $\theta_{\text{grav}} \approx 10^9 / 10^7 = 100$ oscillations more than the lower beam. This phase shift can be observed by the interference pattern of the recombined beams.

In the actual experiment, side effects have to be taken care of: Gravity produces distortions in the single crystal. Contributions from this can be eliminated by comparing X-ray and neutron interference patterns in the same interferometer. Moreover,

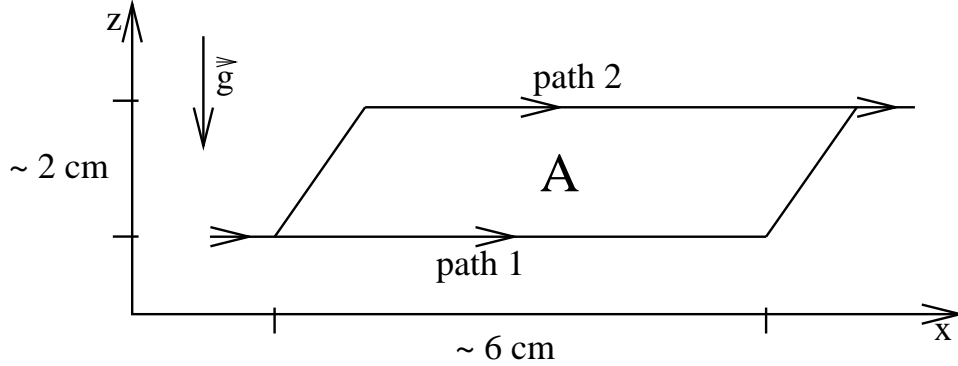


Fig. 3. COW experiment schematically.

the neutron beam itself is bent into a parabolic path with $4 \times 10^{-7} \text{ cm}$ loss in altitude. This yields, however, no significant influence on the phase.

In the COW experiment, the single-crystal interferometer is at rest with respect to the laboratory, whereas the neutrons are subject to the gravitational potential. In order to compare this with the effect of acceleration relative to the laboratory frame, **Bonse and Wroblewski**⁸ let the interferometer oscillate horizontally by driving it via a pair of standard loudspeaker magnets. Thus these experiments of BW and COW test the effect of local acceleration and local gravity on matter waves and prove its *equivalence* up to an accuracy of about 4%.

3.2. Accelerated and rotating reference frame

In order to be able to describe the interferometer in an accelerated frame, we first have to construct a non-inertial frame of reference. If we consider only *mass points*, then a non-inertial frame in the Minkowski space of SR is represented by a *curvilinear* coordinate system, as recognized by Einstein¹³. Einstein even uses the names ‘curvilinear co-ordinate system’ and ‘non-inertial system’ interchangeably.

According to the standard gauge model of electro-weak and strong interactions, a neutron is not a fundamental particle, but consists of *one* up and *two* down quarks which are kept together via the virtual exchange of gluons, the vector bosons of quantum chromodynamics, in a permanent ‘confinement phase’. For studying the properties of the neutron in a non-inertial frame and in low-energy gravity, we may disregard its extension of about 0.7 fm , its form factors, etc. In fact, for our purpose, it is sufficient to treat it as a Dirac particle which carries spin $1/2$ but is *structureless otherwise*.

Table 1. Einstein's approach to GR as compared to the gauge approach: Used are a mass point m or a Dirac matter field Ψ (referred to a local frame), respectively. IF means inertial frame, NIF non-inertial frame. The table refers to special relativity up to the second boldface horizontal line. Below, gravity will be switched on. Note that for the Dirac spinor already the force-free motion in an *inertial* frame *does depend* on the mass parameter m .

	Einstein's approach	gauge approach (\rightarrow COW)
elementary object in SR	mass point m	Dirac spinor $\Psi(x)$
inertial frame	Cartesian coord. system x^i $ds^2 \stackrel{*}{=} o_{ij} dx^i dx^j$	holonomic orthon. frame $e_\alpha = \delta_\alpha^i \partial_i, \quad e_\alpha \cdot e_\beta = o_{\alpha\beta}$
force-free motion in IF	$\dot{u}^i \stackrel{*}{=} 0$	$(i\gamma^i \partial_i - m)\Psi \stackrel{*}{=} 0$
non-inertial frame	arbitrary curvilinear coord. system $x^{i'}$	anholonomic orthon. frame $e_\alpha = e^i{}_\alpha \partial_i$ coframe $\vartheta^\alpha = e_i{}^\alpha dx^i$
force-free motion in NIF	$\dot{u}^{i'} + u^{j'} u^{k'} \{ \begin{smallmatrix} i' \\ j'k' \end{smallmatrix} \} = 0$	$[i\gamma^\alpha e^i{}_\alpha (\partial_i + \Gamma_i) - m]\Psi = 0$ $\Gamma_i := \frac{1}{2} \Gamma_i{}^{\beta\gamma} \rho_{\beta\gamma}$ Lorentz
non-inertial objects	$\{ \begin{smallmatrix} i' \\ j'k' \end{smallmatrix} \}$ 40	$\vartheta^\alpha, \quad \Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha}$ 16 + 24
constraints in SR	$\tilde{R}(\partial\{\}, \{\}) = 0$ 20	$T(\partial e, e, \Gamma) = 0, \quad R(\partial\Gamma, \Gamma) = 0$ 24 + 36
global IF	$g_{ij} \stackrel{*}{=} o_{ij}, \quad \{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \} \stackrel{*}{=} 0$	$(e_i{}^\alpha, \Gamma_i{}^{\alpha\beta}) \stackrel{*}{=} (\delta_i^\alpha, 0)$
switch on gravity	$\tilde{R} \neq 0$ <i>Riemann</i>	$T \neq 0, \quad R \neq 0$ <i>Riemann - Cartan</i>
local IF ('Einstein elevator')	$g_{ij} _P \stackrel{*}{=} o_{ij}, \quad \{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \} _P \stackrel{*}{=} 0$	$(e_i{}^\alpha, \Gamma_i{}^{\alpha\beta}) _P \stackrel{*}{=} (\delta_i^\alpha, 0)$
field equations	$\tilde{Ric} - \frac{1}{2} tr(\tilde{Ric}) \sim mass$ GR	$Ric - \frac{1}{2} tr(Ric) \sim mass$ $Tor + 2 tr(Tor) \sim spin$ EC

A Dirac particle has to be described by means of a four-component Dirac *spinor*. And this spinor is a half-integer representation of the (covering group $SL(2, C)$ of the) Lorentz group $SO(1, 3)$. Therefore at any one point of spacetime we need an *orthonormal reference frame* in order to be able to describe the spinor. Thus, as soon as matter fields are to be represented in spacetime, the notion of a reference system has to be generalized from Einstein's curvilinear coordinate frame ∂_i to an arbitrary, in general anholonomic, orthonormal frame e_α , with $e_\alpha \cdot e_\beta = o_{\alpha\beta}$.

It is possible, of course, to introduce in the Riemannian spacetime of GR arbitrary orthonormal frames, too. However, in the heuristic process of setting up the *fundamental structure* of GR, Einstein and his followers (for a recent example, see the excellent text of d'Inverno³⁶, Secs.9 and 10) restricted themselves to the discussion of mass points and holonomic (natural) frames. Matter waves and arbitrary frames are taboo in this discussion. In Table 1, in the middle column, we displayed the Einsteinian method. Conventionally, *after* the Riemannian spacetime has been found and the dust settled, then electrons and neutron and what not, and their corresponding wave equations, are allowed to enter the scene. But before, they are ignored. This goes so far that the well-documented experiments of COW (1975) and BL (1983) – in contrast to the folkloric Galileo experiments from the leaning tower – seemingly are not even mentioned in d'Inverno³⁶ (1992).

Prugovečki⁷⁹, one of the lecturers here in Erice at our school, in his discussion of the classical equivalence principle, recognizes the decisive importance of orthonormal frames (see his page 52). However, in the end, within his 'quantum general relativity' framework, the good old Levi-Civita connection is singled out again (see his page 125). This is perhaps not surprising, since he considers only zero spin states in this context.

We hope that you are convinced by now that we should introduce arbitrary orthonormal frames in SR in order to represent non-inertial reference systems for matter waves – and that this is important for the setting up of a gravitational gauge theory^{2,42}. The introduction of accelerated observers and thus of non-inertial frames is somewhat standard, even if during the Erice school one of the lecturers argued that those frames are inadmissible. Take the text of Misner, Thorne, and Wheeler⁵⁷. In their Sec.6, you will find an appropriate discussion. Together with Ni³⁰ and in our Honnef lectures²⁷ we tailored it for our needs.

Suppose in SR a non-inertial observer locally measures, by means of the instruments available to him, a three-acceleration \mathbf{a} and a three-angular velocity $\boldsymbol{\omega}$. If the *laboratory coordinates* of the observer are denoted by $x^{\bar{i}}$, with $\bar{\mathbf{x}}$ as the corresponding three-radius vector, then the non-inertial frame can be written in the succinct form^{30,27}

$$\begin{aligned} e_{\hat{0}} &= \frac{1}{1 + \mathbf{a} \cdot \bar{\mathbf{x}}/c^2} \left[\partial_{\bar{0}} - \left(\frac{\boldsymbol{\omega}}{c} \times \bar{\mathbf{x}} \right)^{\bar{B}} \partial_{\bar{B}} \right], \\ e_A &= \partial_{\bar{A}}. \end{aligned} \tag{25}$$

Here ‘naked’ capital Latin letters, $A, \dots = \hat{1}, \hat{2}, \hat{3}$, denote spatial anholonomic components. For completeness we also display the coframe, that is, the one-form basis, which one finds by inverting the frame (25):

$$\begin{aligned}\vartheta^{\hat{0}} &= \left(1 + \frac{\mathbf{a} \cdot \bar{\mathbf{x}}}{c^2}\right) dx^{\bar{0}} = N dx^{\bar{0}}, \\ \vartheta^A &= dx^{\bar{A}} + \left(\frac{\boldsymbol{\omega}}{c} \times \bar{\mathbf{x}}\right)^{\bar{A}} dx^{\bar{0}} = dx^{\bar{A}} + N^{\bar{A}} dx^{\bar{0}}.\end{aligned}\quad (26)$$

In the $(3+1)$ -decomposition of spacetime, N and $N^{\bar{A}}$ are known as *lapse function* and *shift vector*, respectively.

Starting with the coframe, we can read off the connection coefficients (since torsion vanishes in SR) by using Cartan’s first structure equation $d\vartheta^\alpha = -\Gamma_\beta^\alpha \wedge \vartheta^\beta$ with $\Gamma_{\beta}^\alpha = \Gamma_{\bar{i}\beta}^\alpha dx^{\bar{i}}$. We find:

$$\begin{aligned}\Gamma_{\bar{0}\hat{0}A} &= -\Gamma_{\bar{0}A\hat{0}} = \frac{a_A}{c^2}, \\ \Gamma_{\bar{0}AB} &= -\Gamma_{\bar{0}BA} = \epsilon_{ABC} \frac{\omega^C}{c}.\end{aligned}\quad (27)$$

It is important to note that the first index is holonomic, whereas the second and the third indices are anholonomic. If we transform the first index, by means of the frame coefficients $e^{\bar{i}}_\alpha$, with $e_\alpha = e^{\bar{i}}_\alpha \partial_{\bar{i}}$, into an anholonomic one, then we find the totally anholonomic connection coefficients as follows:

$$\begin{aligned}\Gamma_{\hat{0}\hat{0}A} &= -\Gamma_{\hat{0}A\hat{0}} = \frac{a_A}{c^2} / \left(1 + \mathbf{a} \cdot \bar{\mathbf{x}}/c^2\right), \\ \Gamma_{\hat{0}AB} &= -\Gamma_{\hat{0}BA} = \epsilon_{ABC} \frac{\omega^C}{c} / \left(1 + \mathbf{a} \cdot \bar{\mathbf{x}}/c^2\right).\end{aligned}\quad (28)$$

These connection coefficients (28) will enter the Dirac equation referred to a non-inertial frame.

In order to assure ourselves that we didn’t make mistakes in computing the ‘non-inertial’ connection (27,28) by hand, we used for checking its correctness the EXCALC package on exterior differential forms of the computer algebra system REDUCE, see Puntigam et al.⁸⁰ and the literature given there.

3.3. Dirac matter waves in a non-inertial frame of reference

The phase shift (24) can be derived from the Schrödinger equation with a Hamilton operator for a point particle in an external Newton potential. For setting up a gravitational theory, however, one better starts more generally in the special relativistic domain. Thus we have to begin with the Dirac equation in an external gravitational field or, if we expect the equivalence principle to be valid, with the Dirac equation in an accelerated and rotating, that is, in a non-inertial frame of reference.

Take the Minkowski spacetime of SR. Specify Cartesian coordinates. Then the field equation for a massive fermion of spin 1/2 is represented by the Dirac equation

$$i\hbar\gamma^i\partial_i\psi \stackrel{*}{=} mc\psi, \quad (29)$$

where the Dirac matrices γ^i fulfill the relation

$$\gamma^i\gamma^j + \gamma^j\gamma^i = 2\delta^{ij}. \quad (30)$$

For the conventions and the representation of the γ 's, we essentially follow Bjorken-Drell⁷.

Now we straightforwardly transform this equation from an inertial to an accelerated and rotating frame. By analogy with the equation of motion in an arbitrary frame as well as from gauge theory, we can infer the result of this transformation: In the non-inertial frame, the partial derivative in the Dirac equation is simply replaced by the *covariant derivative*

$$\partial_i \Rightarrow D_\alpha := \partial_\alpha + \frac{i}{4}\sigma^{\beta\gamma}\Gamma_{\alpha\beta\gamma}, \quad \partial_\alpha := e^i{}_\alpha\partial_i \equiv e_\alpha, \quad (31)$$

where $\Gamma_{\alpha\beta\gamma}$ are the anholonomic components of the connection, see (28), and x^i the Cartesian coordinates of the lab system (which we called \bar{x}^i previously; we drop the bar for convenience). The anholonomic Dirac matrices are defined by

$$\gamma^\alpha := e_i{}^\alpha\gamma^i \Rightarrow \gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\delta^{\alpha\beta}. \quad (32)$$

The six matrices $\sigma^{\beta\gamma}$ are the infinitesimal generators of the Lorentz group and fulfill the commutation relation

$$[\gamma^\alpha, \sigma^{\beta\gamma}] = 2i(\delta^{\alpha\beta}\gamma^\gamma - \delta^{\alpha\gamma}\gamma^\beta). \quad (33)$$

For Dirac spinors, the Lorentz generators can be represented by

$$\sigma^{\beta\gamma} := (i/2)(\gamma^\beta\gamma^\gamma - \gamma^\gamma\gamma^\beta), \quad (34)$$

furthermore,

$$\boldsymbol{\alpha} := \gamma^0\boldsymbol{\gamma} \quad \text{with} \quad \boldsymbol{\gamma} = \{\gamma^\Xi\}. \quad (35)$$

Then, the Dirac equation, formulated in the orthonormal frame of the accelerated and rotating observer, reads

$$i\hbar\gamma^\alpha D_\alpha\psi = mc\psi. \quad (36)$$

Although there appears now a ‘minimal coupling’ to the connection, which is caused by the change of frame, there is no new physical concept involved in this equation. Only for the measuring devices in the non-inertial frame we have to assume hypotheses similar to the clock hypothesis. This proviso can always be met by a suitable construction and selection of the devices. Since we are still in SR, torsion and curvature

of spacetime both remain zero. Thus (36) is just a reformulation of the ‘Cartesian’ Dirac equation (29).

The rewriting in terms of the covariant derivative provides us with a rather elegant way of explicitly calculating the Dirac equation in the non-inertial frame of an accelerated, rotating observer: Using the anholonomic connection components of (28) as well as $\boldsymbol{\alpha} = -i\{\sigma^{\hat{0}\Xi}\}$, we find for the covariant derivative:

$$\begin{aligned} D_{\hat{0}} &= \frac{1}{1 + \mathbf{a} \cdot \mathbf{x}/c^2} \left(\partial_0 + \frac{1}{2c^2} \mathbf{a} \cdot \boldsymbol{\alpha} - \frac{i}{c\hbar} \boldsymbol{\omega} \cdot \mathbf{J} \right), \\ D_{\Xi} &= \partial_{\Xi}. \end{aligned} \quad (37)$$

The total three-angular momentum operator

$$\mathbf{J} := \mathbf{L} + \mathbf{S} = \mathbf{x} \times \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}} + \frac{1}{2} \hbar \boldsymbol{\sigma} = \mathbf{x} \times \mathbf{p} + \frac{1}{2} \hbar \boldsymbol{\sigma} \quad (38)$$

is composed, in the canonical manner, from the orbital piece \mathbf{L} and the spin piece \mathbf{S} .

The physical effects in our lab frame can be most easily understood by going over to the Hamiltonian. After multiplying the Dirac equation by $\beta := \gamma^{\hat{0}}$ and $c(1 + \mathbf{a} \cdot \mathbf{x}/c^2)$, we get

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi \quad \text{with } H = \beta mc^2 + \mathcal{O} + \mathcal{E}. \quad (39)$$

After substituting the covariant derivatives, the operators \mathcal{O} and \mathcal{E} , which are odd and even with respect to β , read, respectively³⁰:

$$\mathcal{O} := c\boldsymbol{\alpha} \cdot \mathbf{p} + \frac{1}{2c} \{(\mathbf{a} \cdot \mathbf{x})(\mathbf{p} \cdot \boldsymbol{\alpha}) + (\mathbf{p} \cdot \boldsymbol{\alpha})(\mathbf{a} \cdot \mathbf{x})\}, \quad (40)$$

$$\mathcal{E} := \beta m(\mathbf{a} \cdot \mathbf{x}) - \boldsymbol{\omega} \cdot (\mathbf{L} + \mathbf{S}). \quad (41)$$

Up to now, these are mathematically *exact* results. For later purposes we introduce $\mathcal{O}_1 = c\boldsymbol{\alpha} \cdot \mathbf{p}$ and $\mathcal{O}_2 = \mathcal{O} - \mathcal{O}_1$.

Similarly as in quantum electrodynamics⁷, a non-relativistic approximation can be obtained by applying successive Foldy-Wouthuysen transformations. After three such steps we find³⁰, up to the order of c^{-2} ,

$$\begin{aligned} H' &= \beta mc^2 + \frac{\beta}{2m} \mathbf{p}^2 - \frac{\beta}{8m^3 c^2} \mathbf{p}^4 + \beta m(\mathbf{a} \cdot \mathbf{x}) - \boldsymbol{\omega} \cdot (\mathbf{L} + \mathbf{S}) \\ &+ \frac{\beta}{2m} \mathbf{p} \cdot \frac{\mathbf{a} \cdot \mathbf{x}}{c^2} \mathbf{p} + \frac{\beta \hbar}{4mc^2} \boldsymbol{\sigma} \cdot \mathbf{a} \times \mathbf{p} + O\left(\frac{1}{c^3}\right). \end{aligned} \quad (42)$$

The different non-inertial effects of a fermion are displayed in Table 2. Besides the rest mass and the usual kinetic term, we obtain terms which account for the *redshift effect* due to acceleration, thereby verifying the BW and, if the acceleration \mathbf{a} is substituted by the gravitational acceleration \mathbf{g} , the COW experiments. Moreover the

Table 2. Inertial effects for a massive fermion of spin 1/2 in non-relativistic approximation.

$\beta m (\mathbf{a} \cdot \mathbf{x})$	Redshift (Bonse-Wroblewski \rightarrow COW)
$-\boldsymbol{\omega} \cdot \mathbf{L}$	Sagnac type effect (Heer-Werner et al.)
$-\boldsymbol{\omega} \cdot \mathbf{S}$	Spin-rotation effect (Mashhoon)
$\beta \mathbf{p} \cdot (\mathbf{a} \cdot \mathbf{x}) \mathbf{p} / (2mc^2)$	Redshift effect of kinetic energy
$\beta \hbar \boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{p}) / (4mc^2)$	New inertial spin-orbit coupling

‘Sagnac type’ effect occurs in the same manner as in the non-relativistic Schrödinger equation, and a spin-rotation effect is found which, for the neutron interferometer, has first been proposed by Mashhoon⁴⁶. This term could not have been obtained by using the Schrödinger equation.

Thus we demonstrated that the Dirac equation in a Minkowski spacetime,^d referred to a non-inertial frame, yields the BW phase shift and, if the equivalence principle is assumed, also that of the COW experiment. The claim of our colleague in Erice, that in SR such non-inertial frames are illegitimate, is thus disproved. Rather we have shown the usefulness of non-inertial frames ϑ^α , see (26), with $d\vartheta^\alpha \neq 0$.

3.4. ‘Deriving’ a theory of gravity: Einstein’s method as opposed to the gauge procedure

Now we turn back to Table 1: Einstein’s approach, in the middle column, is compared with the gauge approach represented in the right column. The basic idea is that a mechanical system is considered in an inertial and then in a non-inertial frame. In Einstein’s approach the object under investigation is a point particle and the non-inertial frame a *natural* (or coordinate) coframe dx^i , in the gauge approach the object is a Dirac wave function and the non-inertial frame an *arbitrary* (or anholonomic) coframe ϑ^α , with $d\vartheta^\alpha \neq 0$. Einstein’s ‘trick’ was to search for the fields emerging in the non-inertial frame and to identify them as pseudo-gravitational. ‘Pseudo’, since the spacetime is a flat and uncurved Minkowski space without gravitational fields present. This implies that the pseudo-gravitational fields have to obey some constraints, otherwise it wouldn’t be guaranteed that the Minkowski background prevails.

^dThese considerations can be generalized to a Riemannian spacetime, see Huang³⁴ and the literature quoted there.

The gauge approach had been developed by Utiyama et al. long before BW and COW set up their experiments. These experiments confirmed the appropriateness of the earlier theoretical development. Clearly, in some WKB-limit, compare the Foldy-Wouthuysen technique in Sec. 3.3, the Einsteinian approach can be recovered as a limiting case of the more general gauge approach.

In the Einstein case the pseudo-gravitational field is represented by the Christoffel symbol (of the second kind), in the gauge case by the coframe and the Lorentz-connection. The transition from SR to a spacetime with ‘real’ gravity consists in the *relaxation* of the constraints formulated earlier. We are finding a Riemannian and a Riemann-Cartan space^e, respectively.

But beware! Locally we have to recover the Minkowskian behavior of spacetime: In the Einsteinian free-fall elevator gravity does not show up. In the Riemann space this is technically achieved by means of Riemannian normal coordinates, in the case of the Riemann-Cartan space there exist *normal frames* which realize an ‘anholonomic’ elevator, as has been first pointed out by von der Heyde³¹ and developed by one of us²⁵, by Modanese & Toller⁵⁹, Iliev³⁵, and Hartley²³. In Hartley’s formulation²³, we have the following proposition:

Let M be a manifold with metric g and (arbitrary) metric-compatible connection ∇ (Riemann-Cartan space). For any single point $P \in M$, there exist coordinates $\{x^i\}$ and an orthonormal frame $\{e_\alpha\}$ in a neighborhood of P such that

$$\left\{ \begin{array}{l} e_\alpha = \delta_\alpha^i \partial_{x^i} \\ \Gamma_\alpha^\beta = 0 \end{array} \right\} \quad \text{at } P, \quad (43)$$

where Γ_α^β are the connection 1-forms referred to the frame $\{e_\alpha\}$.

Not too many people know of this theorem, even though it had been proposed by von der Heyde already twenty years ago. Usually, if we talk to relativists about the normal frames in a Riemann-Cartan space, they state that those cannot exist, since torsion, as a tensor, cannot be transformed to zero. In this context it is tacitly assumed that the starting point are Riemannian normal coordinates and the torsion is ‘superimposed’. However, since only a *natural* frame is attached to Riemannian normal coordinates, one is too restrictive in the discussion right from the beginning. And, of course, the curvature is also of tensorial nature – and still Riemannian normal coordinates do exist.

In Table 1 there are two subtleties involved which we want to mention:

In the gauge column we formulated two constraints in SR, namely the vanishing of torsion and that of curvature. One could well wonder whether we have to relax

^eRecently Hammond²² gave a very pronounced and interesting plea in favor of the existence of torsion in nature. Only his attempts to derive torsion from some new potential, we don’t find convincing, since it looks ad hoc to us. Moreover, the coframe is some sort of potential already, see (12) and Table 5 – and why should we multiply the number of potentials beyond necessity?

both constraints at the same time. We could allow for torsion only, thereby ending up with a spacetime carrying a teleparallelism, or, alternatively, we could only admit the emergence of curvature, thereby recovering the Riemannian spacetime of GR. It is perhaps surprising that also in the teleparallelism case, see Secs.4 and 5 below, by a suitable choice of a torsion square Lagrangian, one can arrive at a theory which is equivalent to GR. Nevertheless, there are good reasons for relaxing both constraints: Firstly, by mimicking the Einsteinian procedure of the middle column of Table 1, we cannot see any reason why we should lift only one constraint; secondly, by admitting a Riemann-Cartan space, we can still ‘trivialize’ the gravitational potentials at a point P and its neighborhood, as found in proposition (43), in spite of the presence of torsion and curvature. This implies that a Riemann-Cartan space, if described in terms of suitable coframes, looks locally Minkowskian. Eqs.(43) supply the strongest reason for taking the four-dimensional Riemann-Cartan space seriously as a model for spacetime.

A second subtlety is related to the fact that the minimally coupled Dirac equation (36), in a Riemann-Cartan space, slightly differs from the Euler-Lagrange equation of the minimally coupled Dirac *Lagrangian*. However, this shouldn’t cause headaches: The gauge theoretical set up, see Fig. 1, is so closely linked to the Lagrangian formalism – not to speak of the fundamental importance of the Feynman path integral or of Schwinger’s variational principle – that the *minimal coupling* procedure should be implemented on the *Lagrangian level*.

Eventually concentrating our attention to the last row of Table 1, we read off the field equations of GR and of the EC-theory: They result from the simplest conceivable Lagrangian, namely from the curvature scalar of the Riemann or the Riemann-Cartan spacetime respectively. The EC-theory differs from GR in a very weak spin-contact interaction which is unmeasurable at the present time. In this sense, the EC-theory is a *viable* theory of gravitation.

4. Conserved momentum current, the heuristics of the translation gauge

4.1. Motivation

We have in mind to derive gravity from a symmetry principle. But what is the right symmetry to derive gravity from? As already indicated in Sec.2, we think that gravity stems from translation symmetry. Our motivation is the following:

We start from SR. The invariance of the action $W = \int L_{\text{mat}}(\Psi, d\Psi)$ of an isolated material system under rigid spacetime translations yields, by the application of the Noether theorem, a *conserved energy-momentum current* three-form \mathcal{T}_j via

$$\mathcal{T}_j := \frac{\delta L_{\text{mat}}}{\delta dx^j} = \frac{1}{3!} \mathcal{T}_{klmj} dx^k \wedge dx^l \wedge dx^m, \quad d\mathcal{T}_j = 0. \quad (44)$$

(One obtains the “usual” energy-momentum tensor T_{ij} from \mathcal{T}_j by means of $T_{ij} = \epsilon_i^{klm} \mathcal{T}_{klmj}$.) The corresponding charge $M := \int d^3x \mathcal{T}_0$ is conserved in time. In other words: Rigid translational invariance is attributed to the classical field-theoretical equivalent to mass(-energy density), which is the source of Newton-Einstein gravity.

The analogy to electrodynamics guides us of how to actually generate the gravitational interaction. In electrodynamics one finds from rigid $U(1)$ -invariance of an action $W = \int L_{\text{mat}}(\Psi, d\Psi)$ a conserved electromagnetic current with corresponding electric charge Q . As we discussed in length in Sec.2, it is possible to generate the electromagnetic interaction by gauging the rigid $U(1)$ -symmetry. Following this example, we expect the gravitational interaction to emerge from gauging the rigid translational symmetry. To quote Feynman¹⁵: “...gravity is that field which corresponds to a gauge invariance with respect to displacement transformations.” But before we gauge the translation group we should think about...

4.2. Active and passive translations

Whenever we describe quantities with respect to some reference system (a certain basis, a certain coordinate system) we do encounter the possibility of active or passive transformations. Actively transforming a quantity or passively transforming its reference system will usually change the labels of the quantity which are related to the reference system. How can we decide whether an active, a passive, or even a mixture of both kinds of transformation took place when those labels of a quantity changed? Is it necessary to distinguish between active and passive transformations? For the case of active and passive translations, we will answer this question to the extent that we show the mathematical equivalence of actively or passively translating a matter field Ψ in Minkowski spacetime. From this we will later proceed in order to gauge the translation group.

The field Ψ is thought to be some p -form, possibly Lie-algebra valued. We discard any internal structure of Ψ since we are only interested in its spacetime properties. For the Minkowski spacetime we assume for convenience a gauge in which the connection components are identically zero, $\Gamma \stackrel{*}{=} 0$. This can be achieved by choosing pseudo-Cartesian coordinates x^i and using the corresponding holonomic (co-)frame. The total variation $\delta_t \Psi$ of Ψ under translations is the sum of both active and passive variations:

$$\delta_t \Psi = \delta_a \Psi + \delta_p \Psi. \quad (45)$$

An active translation of Ψ is generated by the flow of a vector field $\xi = \xi^i \partial_i$, that is, $\Psi \rightarrow \xi^* \Psi$. In this case the active variation is given by the Lie-derivative l_ξ ,

$$\delta_a \Psi = l_\xi \Psi, \quad l_\xi = d(\xi \rfloor) + (\xi \rfloor) d. \quad (46)$$

Despite possible exterior indices of Ψ , we don't have to use a ‘covariant Lie derivative’ since $\Gamma \stackrel{*}{=} 0$. We will be interested only in infinitesimal variations. Then the Lie

derivative describes the difference between the actively translated $\xi^* \Psi(x^i)$ and $\Psi(x^i + \xi^i)$.

Passive translations are generated by coordinate transformations. To find the passive transformation which corresponds to a given active transformation, we have to know the explicit form of the operator δ_p and the actual coordinate transformation which corresponds to a specific $\delta_a = l_\xi$. For the actual coordinate transformation, we look at the total variation of the coordinate functions x^i ,

$$\delta_t x^i = \delta_a x^i + \delta_p x^i = l_\xi x^i + \delta_p x^i = \xi^i + \delta_p x^i. \quad (47)$$

The coordinate transformation $\delta x^i = \delta_p x^i = -\xi^i$ makes $\delta_t x^i$ to vanish. Thus the passive translation $\delta_p x = \xi$ is equivalent to $\delta_a x = l_\xi x$ (what we intuitively expected without any calculation). Having this candidate for the appropriate coordinate transformation, we look for an operator δ_p such that $\delta_t \Psi = l_\xi \Psi + (\delta_p \Psi)(-\xi) = 0$. We set

$$(\delta_p \Psi)(-\xi) = \Psi(x^i - \xi^i) - \Psi(x^i), \quad (48)$$

i.e., we take the value of Ψ at a point in the coordinate system $x^i - \xi^i$ and subtract the value of Ψ at the same point while using the coordinate system x^i . Writing

$$(\delta_p \Psi)(-\xi) = \Psi_{i_1 \dots i_p}(x^i - \xi^i) d(x^{i_1} - \xi^{i_1}) \wedge \dots \wedge d(x^{i_p} - \xi^{i_p}), \quad (49)$$

Taylor expanding $\Psi_{i_1 \dots i_p}(x^i - \xi^i)$, and keeping terms up to order ξ , we can show that

$$(\delta_p \Psi)(-\xi) = -l_\xi \Psi. \quad (50)$$

Thereby we just rederived the definition of the Lie derivative from a passive point of view, and the coordinate transformation $\delta x = \xi$ becomes clearly equivalent to an active translation generated by l_ξ .

4.3. Heuristic scheme of translational gauging

In order to gauge the translation group we will follow the general gauge scheme which we set up in Sec.2. Therefore we will begin with a rigidly translation invariant action $W = \int L_{\text{mat}}(\Psi, d\Psi)$. We expect to have to introduce four gauge potentials A^α in order to extend from rigid to soft translation invariance of W . Each A^α will compensate one of the four independent soft translations l_{ξ_α} .

We note that soft translations do not commute any longer: Expanding the flow-generating vectors ξ_α according to $\xi_\alpha = e^i{}_\alpha \partial_i$, with spacetime dependent functions $e^i{}_\alpha = e^i{}_\alpha(x)$, we can use the formula

$$[l_{\xi_\alpha}, l_{\xi_\beta}] = l_{[\xi_\alpha, \xi_\beta]} \quad (51)$$

to show that (the equality $\stackrel{*}{=}$ requires the gauge $\Gamma \stackrel{*}{=} 0$)

$$[l_{\xi_\alpha}, l_{\xi_\beta}] = -e^j{}_{[\alpha} e^i{}_{\beta]} e_i{}^\gamma{}_{,j} \xi_\gamma \stackrel{*}{=} -T_{\alpha\beta}{}^\gamma \xi_\gamma. \quad (52)$$

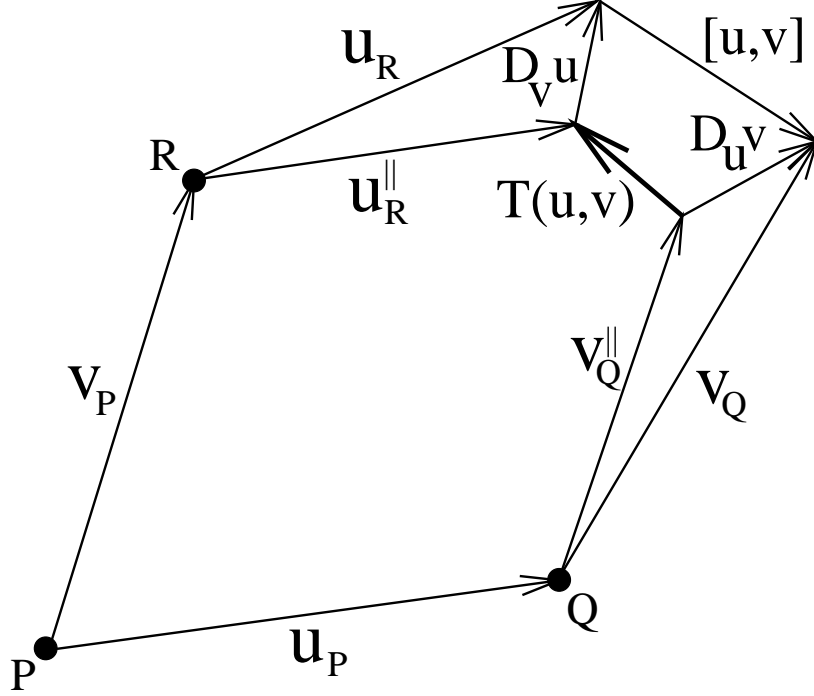


Fig. 4. On the geometrical interpretation of torsion: It represents a closure failure of infinitesimal displacements.

The $T_{\alpha\beta}{}^\gamma$ are the components of the torsion two-form $T^\gamma = \frac{1}{2} T_{\alpha\beta}{}^\gamma \vartheta^\alpha \wedge \vartheta^\beta$ expanded in the anholonomic coframe $\vartheta^\alpha = e_i^\alpha dx^i$. This softening of the Lie algebra of the translation group should be seen in correspondence to the structure

$$\text{commutator of soft symmetry transformations} = \text{field strength}, \quad (53)$$

which is well known from Yang-Mills theory. Therefore it is tempting to view the torsion tensor as the translational field strength.

The torsion tensor represents a translational misfit, as already indicated in Sec.2.4. A more detailed discussion is given by de Sabbata and Sivaram⁸² who also collected numerous other facts and results on torsion. Torsion measures the noncommutativity of displacements of points in analogy to the curvature tensor which measures the noncommutativity of displacements of vectors. This is explained in Fig. 4: Let P be a point of the (spacetime) manifold M and v_P, u_P two linearly independent vectors of $T_P M$. We regard v_P, u_P as infinitesimally small. Then they define two points, R and Q , on M . (We can make this mathematically precise by defining R, Q via the exponential map $\exp_P : T_P M \rightarrow M$, $\exp_P(v) = R$, $\exp_P(u) = Q$.) In other words: The vector v_P displaces P infinitesimally to R , the vector u_P displaces P infinitesimally to Q . The prescription to perform two successive displacements v, u is to first displace P by means of v_P to R and second parallel transport u_P to $u_R^||$.

and displace R by means of u_R^\parallel . Fig. 4 shows that the commutator of two successive displacements won't vanish in general – the gap between the two resulting points is, by definition, a measure for the torsion. It is also shown in Fig.4 that this definition matches the usual textbook definition

$$T(u, v) := D_u v - D_v u - [u, v]. \quad (54)$$

To make contact with our present notation, we evaluate the components of T with respect to an arbitrary frame e_α :

$$\begin{aligned} T(e_\beta, e_\gamma) &= D_{e_\beta} e_\gamma - D_{e_\gamma} e_\beta - [e_\beta, e_\gamma] \\ &= \Gamma_{\beta\gamma}^\alpha e_\alpha - \Gamma_{\gamma\beta}^\alpha e_\alpha + C_{\beta\gamma}^\alpha e_\alpha \\ &= T_{\beta\gamma}^\alpha e_\alpha. \end{aligned} \quad (55)$$

Hence

$$T_{\beta\gamma}^\alpha = C_{\beta\gamma}^\alpha + \Gamma_{\beta\gamma}^\alpha - \Gamma_{\gamma\beta}^\alpha, \quad (56)$$

or, by using the object of anholonomy

$$C^\alpha := d\vartheta^\alpha = \frac{1}{2} C_{\beta\gamma}^\alpha \vartheta^\beta \wedge \vartheta^\gamma, \quad (57)$$

we find:

$$T^\alpha = d\vartheta^\alpha + \Gamma_\beta^\alpha \wedge \vartheta^\beta. \quad (58)$$

The three irreducible pieces of the torsion under $SO(1, 3)$ -decomposition are displayed for later use in Tab.3.

Table 3. Irreducible decomposition of the torsion $T^\alpha = {}^{(1)}T^\alpha + {}^{(2)}T^\alpha + {}^{(3)}T^\alpha$ under the Lorentz group $SO(1, 3)$

	explicit expression	number of components	name
${}^{(2)}T^\alpha$	$\frac{1}{3}\vartheta^\alpha \wedge (e_\beta]T^\beta)$	4	TRATOR
${}^{(3)}T^\alpha$	$-\frac{1}{3}*(\vartheta^\alpha \wedge *(T^\beta \wedge \vartheta_\beta))$	4	AXITOR
${}^{(1)}T^\alpha$	$T^\alpha - {}^{(2)}T^\alpha - {}^{(3)}T^\alpha$	16	TENTOR

From the translational field strength T^α , see (52) and (53), we come back to the translational gauge potential A^α . According to the general gauge scheme, it will

couple to the energy-momentum current according to

$$\mathcal{T}_\alpha = \frac{\delta L}{\delta A^\alpha}. \quad (59)$$

The left hand side of (44) suggests that the gauge process will replace the holonomic coframe dx^i by the potentials A^α . Do the A^α have something to do with an anholonomic coframe ϑ^α ? A consistent relation between T^α , ϑ^α , and A^α is established if we assume that

$$\vartheta^\alpha = \delta_i^\alpha dx^i + A^\alpha. \quad (60)$$

Then we obtain

$$dA^\alpha = d\vartheta^\alpha \stackrel{*}{=} T^\alpha. \quad (61)$$

From (61) we get the ‘Bianchi’ identity $dT^\alpha \stackrel{*}{=} 0$. Comparison to the general Bianchi identity $DT^\alpha = R_\beta^\alpha \wedge \vartheta^\beta$ indicates that the structures found so far are part of a more general framework. This is, indeed, the case. In order to derive GR from as little input as possible, we refrain at this point from introducing additional fields, as for example an independent linear connection Γ . We collected the structures relevant for the translational gauge scheme in Tab.4.

Table 4. The relevant structures in a gauge approach of the four parameter translation group

		number of components
conserved momentum current	\mathcal{T}_α	4×4
translational gauge potential	$\vartheta^\alpha \simeq A^\alpha$ (coframe)	4×4
translational field strength	T^α (torsion)	4×6
1st Bianchi identity ($\Gamma \stackrel{*}{=} 0$)	$dT^\alpha \stackrel{*}{=} 0$	4×4

5. Theory of the translation gauge: From Einsteinian teleparallelism to GR

As we argued in the last section, GR should be derivable from gauge ideas in a fairly straightforward manner. The key to arrive at GR is to start with the conserved momentum current and to gauge the translation group that is connected with it.

5.1. Translation gauge potential

We commence with the very basics: Consider again a field theory in Minkowski spacetime (pseudo-Cartesian coordinates) defined by the Lagrangian $L_{\text{mat}} = L_{\text{mat}}(\Psi, d\Psi)$.

An explicit dependence of L on the coordinates x^i is already forbidden by rigid translational invariance, which we started from. The coordinates enter more implicitly: The field Ψ has to be expressed in terms of differential forms in order to build the Lagrangian 4-form L as the appropriate integrand of the action. Therefore one needs the differentials dx^i as natural (or holonomic) basis for the physical field. They are invariant under rigid but not under soft translations:

$$\delta x^i = \varepsilon^i(x) \implies \delta dx^i = d\varepsilon^i. \quad (62)$$

Referring to the equivalence between active and passive transformations, which is valid at this stage (see Sec.4.2), we view in this approach the translations as passive transformations. The differentials dx^i are no longer sufficient to build up an invariant Lagrangian. According to the gauge principle, we have to introduce a gauge potential A^α with transformation behavior $\delta A^\alpha = -\delta_i^\alpha d\varepsilon^i$ such that

$$\vartheta^\alpha := \delta_i^\alpha \hat{D}x^i := \delta_i^\alpha dx^i + A^\alpha \quad (63)$$

transforms like

$$\delta \vartheta^\alpha = 0. \quad (64)$$

The anholonomic one-form basis ϑ^α serves as an appropriate form basis for Lagrangians L_{mat} since it automatically incorporates soft translational invariance. One usually refers to ϑ^α , instead of A^α , as the translational potential. Normally, a distinction between ϑ^α and A^α is not necessary, since their field components differ just by a constant (1 or 0).

5.2. Lagrangian

The corresponding field strength $T^\alpha \stackrel{*}{=} d\vartheta^\alpha$ can be used to construct a kinematic supplementary term for ϑ^α to the Lagrangian. The double role of ϑ^α as both, a dynamical gauge potential and an orthonormal frame (defining a new metric via $g = o_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta$), explains the transition from Minkowski space to a dynamical spacetime, which is due to translational invariance. For the kinematic term we make the quadratic ansatz $V = d\vartheta^\alpha \wedge H_\alpha$. What would be a good choice for H_α ? Eyeing at Yang-Mills theory, we are tempted to put $H_\alpha = \frac{1}{2\ell^2} *d\vartheta_\alpha$, with $\ell = \text{Planck length}$. But we would like to end up with a softly Lorentz invariant theory. The Lagrangian $V = \frac{1}{2\ell^2} *d\vartheta^\alpha \wedge d\vartheta_\alpha$ is rigidly but not softly Lorentz invariant, though. Note that this postulate of soft Lorentz symmetry is not equivalent to a gauging of the Lorentz group! We won't cure the lack of Lorentz invariance by the introduction of some new gauge field Γ for the Lorentz field, but will use it just as a criterion of choosing a good Lagrangian.

The most general term V quadratic in $d\vartheta^\alpha$ is obtained by choosing H_α as

$$H_\alpha = \frac{1}{2\ell^2} * \left(a_1^{(1)} d\vartheta_\alpha + a_2^{(2)} d\vartheta_\alpha + a_3^{(3)} d\vartheta_\alpha \right), \quad (65)$$

see also Mielke^{50,49}. The pieces ${}^{(I)}d\vartheta^\alpha$ correspond to the irreducible pieces ${}^{(I)}T^\alpha$ of the torsion, compare Table 3:

$$\begin{aligned} {}^{(2)}d\vartheta^\alpha &:= \frac{1}{3}\vartheta^\alpha \wedge (e_\beta \rfloor d\vartheta^\beta), \\ {}^{(3)}d\vartheta^\alpha &:= -\frac{1}{3}*\{\vartheta^\alpha \wedge *(d\vartheta^\beta \wedge \vartheta_\beta)\}, \\ {}^{(1)}d\vartheta^\alpha &:= d\vartheta^\alpha - {}^{(2)}d\vartheta^\alpha - {}^{(3)}d\vartheta^\alpha. \end{aligned} \quad (66)$$

The postulate of soft Lorentz invariance leads to a solution for the constant and real parameters a_I in the following way:

Infinitesimal Lorentz rotations are expressed by $\delta\vartheta^\alpha = \varepsilon^\alpha{}_\beta \vartheta^\beta$ where $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$ are the antisymmetric Lorentz group parameters. It is easy to check that the gauge Lagrangian $V = d\vartheta^\alpha \wedge H_\alpha$, with H_α given by (65), is invariant under *rigid* Lorentz rotations, $\delta V = 0$. The general expression for δV reads

$$\delta V = \left(\frac{\partial V}{\partial \vartheta^\alpha} - d \frac{\partial V}{\partial d\vartheta^\alpha} \right) \wedge \delta\vartheta^\alpha + d \left(\frac{\partial V}{\partial d\vartheta^\alpha} \wedge \delta\vartheta^\alpha \right). \quad (67)$$

Hence we have $\delta V = 0$ for rigid Lorentz rotations. However, for *soft* Lorentz rotations with spacetime-dependent group parameters $\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}(x)$, we get from (67) the offending term

$$\delta_{(\text{soft})} V = d\varepsilon^\alpha{}_\beta \wedge \frac{\partial V}{\partial d\vartheta^\alpha} \wedge \vartheta^\beta. \quad (68)$$

In order to preserve Lorentz invariance, this term has to be canceled, modulo an exact form. Using the Leibniz rule, we obtain

$$d\varepsilon^\alpha{}_\beta \wedge \frac{\partial V}{\partial d\vartheta^\alpha} \wedge \vartheta^\beta = \varepsilon^\alpha{}_\beta d \left(\frac{\partial V}{\partial \vartheta^\alpha} \wedge \vartheta^\beta \right) - d \left(\varepsilon^\alpha{}_\beta \frac{\partial V}{\partial \vartheta^\alpha} \wedge \vartheta^\beta \right). \quad (69)$$

The second term on the r.h.s. is already exact. From the first term we get the condition

$$\frac{\partial V}{\partial d\vartheta}{}_{[\alpha} \wedge \vartheta_{\beta]} = \text{exact form} \quad (70)$$

for soft Lorentz invariance of V . We plug in the explicit expression for V and obtain, after some algebra,

$$\begin{aligned} \frac{\partial V}{\partial d\vartheta}{}_{[\alpha} \wedge \vartheta_{\beta]} &= \left(\frac{1}{3}a_1 - \frac{1}{3}a_3 \right) d\eta_{\alpha\beta} - \left(\frac{2}{3}a_3 + \frac{1}{3}a_1 \right) d\vartheta_{[\alpha} \wedge \vartheta_{\beta]} \\ &+ \left(\frac{1}{6}a_1 + \frac{1}{6}a_2 - \frac{1}{3}a_3 \right) (e_\gamma \rfloor d\vartheta^\gamma) \wedge \eta_{\alpha\beta}. \end{aligned} \quad (71)$$

The last two terms can be made vanishing by choosing

$$a_2 = -2a_1, \quad a_3 = -\frac{1}{2}a_1. \quad (72)$$

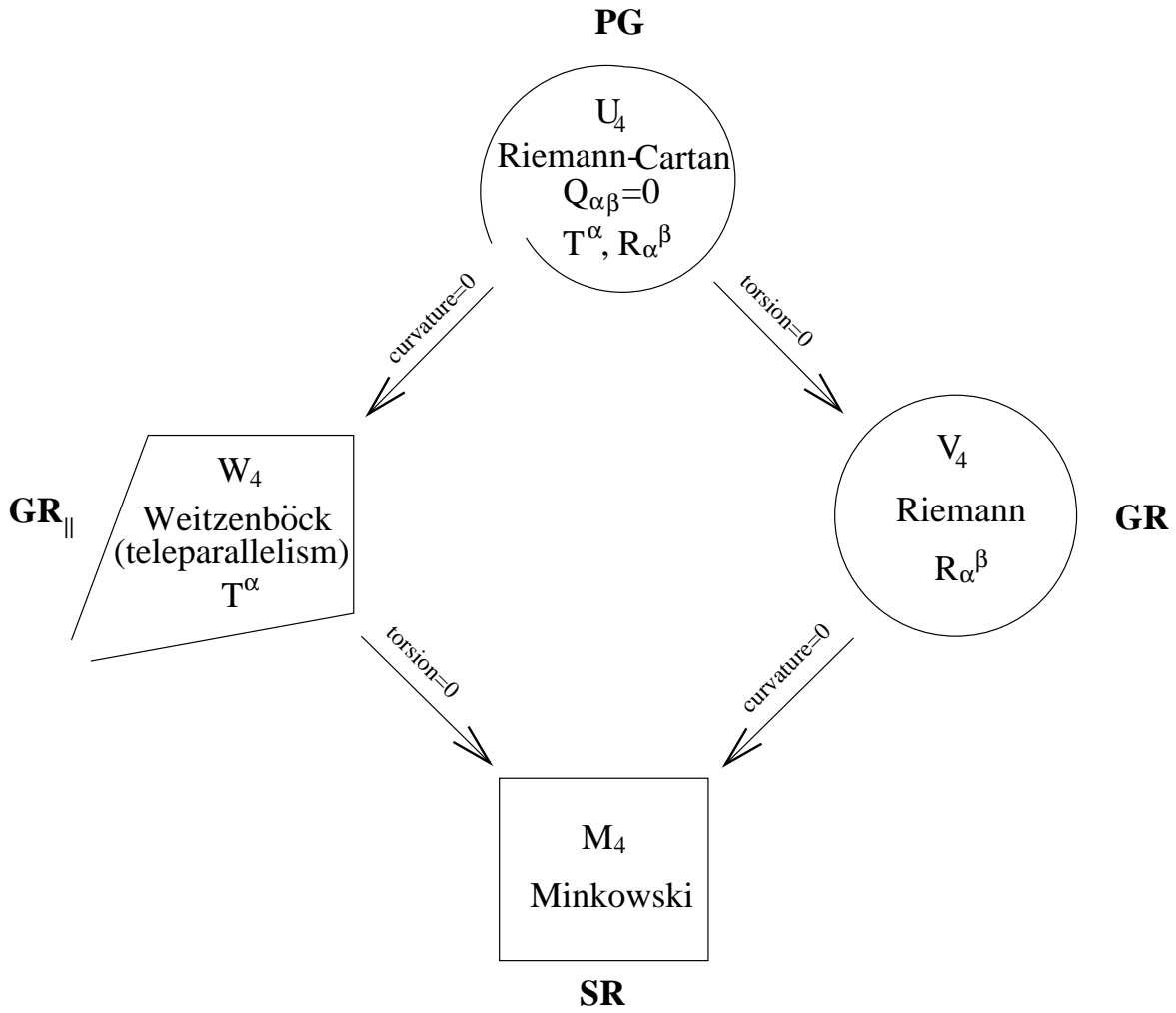


Fig. 5. A spacetime with a metric and a metric compatible connection (nonmetricity=0) is called a Riemann-Cartan space U_4 . It can either become a Weitzenböck space W_4 , if its curvature vanishes, or a Riemann space V_4 , if the torsion happens to vanish. These different models of spacetime are the arenas for different gravitational theories.

Then we obtain

$$\frac{\partial V}{\partial d\vartheta}[\alpha \wedge \vartheta_\beta] = \frac{a_1}{2} d\eta_{\alpha\beta}. \quad (73)$$

The constant a_1 can be absorbed by a suitable choice of the coupling constant ℓ in V , see (65). According to the usual conventions, we put $a_1 = -1$, i.e. V is softly Lorentz invariant for the choice of parameters

$$a_1 = -1, \quad a_2 = 2, \quad a_3 = \frac{1}{2}. \quad (74)$$

Hence

$$V_{||} = \frac{1}{2\ell^2} d\vartheta^\alpha \wedge * \left(-^{(1)}d\vartheta_\alpha + 2^{(2)}d\vartheta_\alpha + \frac{1}{2}^{(3)}d\vartheta_\alpha \right). \quad (75)$$

The total Lagrangian reads

$$L_{\text{tot}} = V_{||} + L_{\text{mat}}(\Psi, d\Psi, \vartheta^\alpha), \quad (76)$$

and the field equation $\delta L_{\text{tot}}/\delta\vartheta^\alpha = 0$ becomes

$$dH_\alpha - E_\alpha = \mathcal{T}_\alpha, \quad (77)$$

where, as before, $\mathcal{T}_\alpha = \delta L_{\text{mat}}/\delta\vartheta^\alpha$ denotes the canonical energy-momentum three-form and

$$\begin{aligned} E_\alpha &:= (e_\alpha]d\vartheta^\beta) \wedge H_\beta - \frac{1}{2} e_\alpha](d\vartheta^\beta \wedge H_\beta) \\ &= \frac{1}{2} [(e_\alpha]d\vartheta^\beta) \wedge H_\beta - d\vartheta^\beta \wedge (e_\alpha]H_\beta), \end{aligned} \quad (78)$$

the energy-momentum current of the gauge field.

5.3. Transition to GR

If the Lagrangian (75) is substituted into the field equation (77), then it can be seen that the antisymmetric piece of the left hand side of (77) vanishes,

$$\vartheta_{[\beta} \wedge dH_{\alpha]} - \vartheta_{[\beta} \wedge E_{\alpha]} = 0. \quad (79)$$

Therefore the right hand side has to be symmetric, too, i.e. only *scalar* matter fields or *gauge* fields, such as the electromagnetic field, are allowed as material sources, whereas matter carrying spin cannot be consistently coupled in such a framework. The existing nontrivial torsion, expressed by $d\vartheta^\alpha$, describes the nontrivial Riemannian geometry of spacetime. This is because we have tied the Christoffel connection $\overset{\{\}}{\Gamma}_{\alpha\beta}$, which is determined by the metric, to the contortion tensor $K_{\alpha\beta}$ by means of the teleparallel condition $\Gamma_{\alpha\beta} \stackrel{*}{=} 0$:

$$\overset{\{\}}{\Gamma}_{\alpha\beta} - K_{\alpha\beta} = \Gamma_{\alpha\beta} \stackrel{*}{=} 0 \quad \Longrightarrow \quad \overset{\{\}}{\Gamma}_{\alpha\beta} \stackrel{*}{=} K_{\alpha\beta}. \quad (80)$$

In other words: Meaningful teleparallel theories do *not* presuppose spinning matter as a source for nontrivial torsion, in contrast to what is sometimes stated in literature³⁹.

Doesn't all this look like general relativity? The Levi-Civita (or Christoffel) connection, corresponding to the metric $g = o_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta$, is given by

$$\overset{\{\}}{\Gamma}_{\alpha\beta} = \frac{1}{2} (e_\alpha]d\vartheta_\beta - e_\beta]d\vartheta_\alpha - (e_\alpha]e_\beta]d\vartheta_\gamma) \wedge \vartheta^\gamma). \quad (81)$$

The corresponding Riemannian curvature reads

$$\overset{\{\}}{R}_{\alpha\beta} = d\overset{\{\}}{\Gamma}_{\alpha\beta} - \overset{\{\}}{\Gamma}_{\alpha\gamma} \wedge \overset{\{\}}{\Gamma}^{\gamma}_{\beta}. \quad (82)$$

We use the last two equations to replace on the Lagrangian level the variable $d\vartheta^\alpha$ by $\overset{\{\}}{\Gamma}_{\alpha\beta}$. Using (81) and (82), one can prove the remarkable identity

$$\frac{1}{2} \overset{\{\}}{R}^{\alpha\beta} \wedge \eta_{\alpha\beta} - \ell^2 V = d(\vartheta^\alpha \wedge *d\vartheta_\alpha), \quad (83)$$

with V given by (75). Therefore one finds that the kinematic term V , with the above choice of parameters a_I , is equivalent to the Hilbert-Einstein action modulo an exact term. Replacing V in the action S by means of (83) leads, via $\delta L_{\text{tot}}/\delta\vartheta^\alpha = 0$, to Einstein's equation

$$\overset{\{\}}{G}_\alpha := \frac{1}{2} \eta_{\alpha\beta\gamma} \wedge \overset{\{\}}{R}^{\beta\gamma} = \ell^2 \mathcal{T}_\alpha. \quad (84)$$

In such a way, we arrive at GR in its original form. Shifting back and forth from the variable pair $(\vartheta^\alpha, \overset{\{\}}{\Gamma}_\alpha{}^\beta)$ to $(\vartheta^\alpha, d\vartheta^\alpha)$ means shifting back and forth from original GR to its teleparallel equivalent GR_{\parallel} . This is displayed in Fig.5: A general PG induces a Riemann-Cartan space U_4 with nonvanishing torsion and curvature. Such a U_4 can be reduced to either a Weitzenböck space W_4 (curvature = 0) or a Riemann space V_4 (torsion = 0), i.e. to the geometries induced by GR_{\parallel} or GR, respectively.

We wish to point out that both theories can be obtained as special cases within the framework of the Poincaré gauge theory PG, see our Physics Reports²⁹, the literature given there, and the work of Pascual-Sánchez^{72,73}.

6. Gauging of the affine group $R^4 \rtimes GL(4, R)$

We recognized that the gauging of the translations yields a theory which, for spinless matter and for electromagnetism, that is, for symmetric energy-momentum currents, is equivalent to GR. Thus we have a new understanding of Einstein's theory. Why should we generalize GR_{\parallel} if it is consistent with experiment? Three somewhat interrelated arguments come to mind:

- The translations represent a subgroup of the Poincaré group. Only to gauge the translations and to leave the Lorentz subgroup of the Poincaré group untouched, would seem unnatural. This argument is all the more convincing, since the *semi*-direct product structure of the Poincaré group interrelates its two mentioned subgroups stronger than it were the case for a direct product.
- A Weitzenböck spacetime is a degenerate Riemann-Cartan space. The gauge arguments against the rigidity of a teleparallelism were already advanced by Weyl in the twenties against Einstein's corresponding theory.
- The translational gauge procedure, as it is obvious from the field equation (77) with (75), works only for spinless matter and for electromagnetism, since the field equation is symmetric, see (79), and supplies only 10 independent components.

We conclude that a gauging of the whole 4 + 6 parameter Poincaré group is mandatory. The theory which emerges, the Poincaré gauge theory (PG), is formulated in a Riemann-Cartan spacetime, the Einstein-Cartan theory (EC) being a degenerate subcase of it. And the EC is a viable gravitational theory!

In order to see the built-up of the different structures of spacetime more clearly, we prefer to gauge⁴⁸ immediately the 4 + 16 parameter *affine* group⁴⁸ $A(4, R) = R^4 \rtimes GL(4, R)$ — which lacks a metric structure altogether — and to introduce the metric *subsequently*. Thus we arrive at the metric-affine gauge theory (MAG) which encompasses the PG as a subcase. Symbolically, we may write

$$\text{MAG}/_{\text{nonmetricity}=0} = \text{PG}. \quad (85)$$

The actual gauging of the affine group was done in our Physics Reports²⁹, and we follow the presentation given there. A short outline of these results, see Secs. 3.1, 3.2, 3.3, loc. cit., will be given here. We start then in the flat n -dimensional affine space R^n . The rigid affine group $A(n, R) := R^n \rtimes GL(n, R)$ is the semidirect product of the group of n -dimensional *translations* and n -dimensional *general linear transformations*. This transformation group, cf. ref.⁽⁴⁰⁾ p. 27, acts on an affine n -vector $x = \{x^\alpha\}$ according to

$$x \quad \rightarrow \quad x' = \Lambda x + \tau, \quad (86)$$

where $\Lambda = \{\Lambda^\alpha_\beta\} \in GL(n, R)$ and $\tau = \{\tau^\alpha\} \in R^n$. Thus it is a generalization of the Poincaré group $P := R^4 \rtimes SO(1, 3)$, with the pseudo-orthogonal group $SO(1, n-1)$ being replaced by the general linear group $GL(n, R)$. It is convenient to work with a Möbius type representation for which we take the same symbol $A(n, R)$: It is that subgroup of $GL(n+1, R)$ which leaves the n -dimensional hyperplane $\bar{R}^n := \left\{ \bar{x} = \begin{pmatrix} x \\ 1 \end{pmatrix} \in R^{n+1} \right\}$ invariant:

$$A(n, R) = \left\{ \begin{pmatrix} \Lambda & \tau \\ 0 & 1 \end{pmatrix} \in GL(n+1, R) \mid \Lambda \in GL(n, R), \tau \in R^n \right\}. \quad (87)$$

Thus, by an affine transformation, we obtain

$$\bar{x}' = \begin{pmatrix} \Lambda & \tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda x + \tau \\ 1 \end{pmatrix}, \quad (88)$$

as is required for the action of the affine group on the flat affine space. The Lie algebra $a(n, R)$ consists of the generators P_γ , representing n -dimensional translations, and the L^α_β , which span the Lie algebra $gl(n, R)$ of n -dimensional linear transformations.

In a matrix representation we can write the affine *gauge* group as

$$\mathcal{A}(n, R) = \left\{ \begin{pmatrix} \Lambda(x) & \tau(x) \\ 0 & 1 \end{pmatrix} \mid \Lambda(x) \in \mathcal{GL}(n, R), \tau(x) \in \mathcal{T}(n, R) \right\}. \quad (89)$$

Having already had some experience with the Yang-Mills type gauge approach, we are aware of the need of introducing a gauge potential in order to step from rigid to soft group invariance. Accordingly, by gauging the affine group, the softening of the

affine group transformations is accompanied by the introduction of the *generalized affine* connection (\rightarrow potential)

$$\tilde{\Gamma} = \begin{pmatrix} \Gamma^{(L)} & \Gamma^{(T)} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Gamma_\alpha^{(L)\beta} L^\alpha_\beta & \Gamma^{(T)\alpha} P_\alpha \\ 0 & 0 \end{pmatrix}. \quad (90)$$

It is a one-form $\tilde{\Gamma} = \tilde{\Gamma}_i dx^i$ and transforms inhomogeneously under an affine gauge transformation,

$$\tilde{\Gamma} \xrightarrow{A^{-1}(x)} \tilde{\Gamma}' = A^{-1}(x) \tilde{\Gamma} A(x) + A^{-1}(x) dA(x), \quad A(x) \in \mathcal{A}(n, R), \quad (91)$$

where the transformation is formed with respect to the group element

$$A^{-1}(x) = \begin{pmatrix} \Lambda^{-1}(x) & -\Lambda^{-1}(x)\tau(x) \\ 0 & 1 \end{pmatrix}. \quad (92)$$

The corresponding affine curvature is given by

$$\tilde{R} := d\tilde{\Gamma} + \tilde{\Gamma} \wedge \tilde{\Gamma} = \begin{pmatrix} R^{(L)} & R^{(T)} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} d\Gamma^{(L)} + \Gamma^{(L)} \wedge \Gamma^{(L)} & d\Gamma^{(T)} + \Gamma^{(L)} \wedge \Gamma^{(T)} \\ 0 & 0 \end{pmatrix}. \quad (93)$$

It transforms covariantly under the affine gauge group:

$$\tilde{R} \xrightarrow{A^{-1}(x)} \tilde{R}' = A^{-1}(x) \tilde{R} A(x). \quad (94)$$

The exterior covariant derivative $\tilde{D} := d + \tilde{\Gamma} \wedge$ acts on an affine p -form $\tilde{\Psi} = \begin{pmatrix} \Psi \\ 1 \end{pmatrix}$ as follows:

$$\tilde{D} \tilde{\Psi} = \begin{pmatrix} d\Psi + \Gamma^{(L)} \wedge \Psi + \Gamma^{(T)} \\ 0 \end{pmatrix} = \begin{pmatrix} D\Psi + \Gamma^{(T)} \\ 0 \end{pmatrix}. \quad (95)$$

After substitution of (90) and (92), the inhomogeneous transformation law (91) splits into

$$\Gamma^{(L)} \xrightarrow{A^{-1}(x)} \Gamma^{(L)'} = \Lambda^{-1}(x) \Gamma^{(L)} \Lambda(x) + \Lambda^{-1}(x) d\Lambda(x), \quad (96)$$

and

$$\Gamma^{(T)} \xrightarrow{A^{-1}(x)} \Gamma^{(T)'} = \Lambda^{-1}(x) \Gamma^{(T)} + \Lambda^{-1}(x) D\tau(x). \quad (97)$$

The soft translations $\tau(x)$ automatically drop out in (96) due to the one-form structure of $\Gamma^{(T)}$. Thereby (96) acquires the conventional transformation rule (with the exterior derivative d) of a Yang–Mills–type connection for $\mathcal{GL}(n, R)$. Thus we can identify $\Gamma^{(L)} = \Gamma = \Gamma_\alpha^\beta L^\alpha_\beta$ with the linear connection and are basically done with the gauging of the linear part of the affine group.

Things are more involved for the translational part, though. From what we have learned by gauging the translation group we expect $\Gamma^{(T)}$ to be related to the coframe $\vartheta := \vartheta^\alpha P_\alpha$, i.e. to a one-form with values in the Lie algebra of R^n . But due to the

Table 5. Gauge fields in metric-affine gauge theory MAG

Potential	Field strength	Bianchi identity
metric $g_{\alpha\beta}$	$Q_{\alpha\beta} = -Dg_{\alpha\beta}$	$DQ_{\alpha\beta} = 2R_{(\alpha}{}^{\mu} g_{\beta)\mu}$
coframe ϑ^α	$T^\alpha = D\vartheta^\alpha$	$DT^\alpha = R_\mu{}^\alpha \wedge \vartheta^\mu$
connection $\Gamma_\alpha{}^\beta$	$R_\alpha{}^\beta = d\Gamma_\alpha{}^\beta - \Gamma_\alpha{}^\mu \wedge \Gamma_\mu{}^\beta$	$DR_\alpha{}^\beta = 0$

covariant exterior derivative term $D\tau(x) := d\tau(x) + \Gamma^{(L)}\tau(x)$ in (97), the translational part $\Gamma^{(T)}$ does not transform as a covector, as is required for the coframe ϑ . To get a correspondence between $\Gamma^{(T)}$ and ϑ , we introduce a vector (vector-valued zero-form) $\tilde{\xi} = \begin{pmatrix} \xi \\ 1 \end{pmatrix} = \begin{pmatrix} \xi^\alpha P_\alpha \\ 1 \end{pmatrix}$ which transforms as $\tilde{\xi}' = A^{-1}(x)\tilde{\xi}$, i.e. as

$$\xi \xrightarrow{A^{-1}(x)} \xi' = \Lambda^{-1}(x) (\xi - \tau(x)) \quad (98)$$

under an active affine gauge transformation. Then we define

$$\vartheta := \Gamma^{(T)} + D\xi \quad (99)$$

which transforms as a vector-valued one-form under the $\mathcal{A}(n, R)$, as required:

$$\vartheta \xrightarrow{A^{-1}(x)} \vartheta' = \Lambda^{-1}(x)\vartheta. \quad (100)$$

The geometric and physical meaning of the relation (99), especially the role of the field ξ , is perhaps not completely satisfactorily clarified, yet. For an expanded discussion of this point we refer the reader again to ref.⁽²⁹⁾, see also Mistura⁵⁸. The translational piece of the affine connection, namely $\Gamma^{(T)}$, could play a decisive role in the gravitationally induced phase factor of a matter wave, as was pointed out by Morales-Técotl et al.⁶⁰. Here, we confine ourselves to require the condition

$$D\xi = 0 \quad (101)$$

to hold. Then the generalized affine connection $\tilde{\tilde{\Gamma}}$ on the affine bundle $A(M)$ reduces to the *Cartan connection*

$$\tilde{\tilde{\Gamma}} = \begin{pmatrix} \Gamma^{(L)} & \vartheta \\ 0 & 0 \end{pmatrix} \quad (102)$$

on the bundle $L(M)$ of linear frames. Due to (100), this is not anymore a connection in the usual sense. We are thus left with the potentials $(\vartheta^\alpha, \Gamma_\alpha{}^\beta)$ and their corresponding field strengths $(T^\alpha, R_\alpha{}^\beta)$.

For this gauging of the affine group no metric was necessary. If additionally a *metric* is given, we recover the metric–affine geometrical arena of Sec. 2.4, as is summarized in Table 5.

We close this geometric section by introducing a further geometric ingredient, the “ η -bases”, which span the graded algebra of dual exterior forms on each cotangent space and which turn out to be quite useful in practical calculations. First, owing to the existence of a metric, we can define the scalar density $\sqrt{|\det g_{\mu\nu}|}$ and the familiar Hodge star operator $*$. The Hodge star operator maps a p -form Ψ into an $(n-p)$ -form $*\Psi$ by means of the explicit formula

$$*\Psi := \frac{1}{(n-p)!p!} \sqrt{|\det g_{\mu\nu}|} g^{\alpha_1\gamma_1} \dots g^{\alpha_p\gamma_p} \epsilon_{\alpha_1\dots\alpha_p\beta_1\dots\beta_{n-p}} \Psi_{\gamma_1\dots\gamma_p} \vartheta^{\beta_1} \wedge \dots \wedge \vartheta^{\beta_{n-p}}. \quad (103)$$

Now we define the g -volume element n -form

$$\eta := \sqrt{|\det g_{\mu\nu}|} \vartheta^{\hat{1}} \wedge \dots \wedge \vartheta^{\hat{n}} = \frac{1}{n!} \sqrt{|\det g_{\mu\nu}|} \epsilon_{\alpha_1\dots\alpha_n} \vartheta^{\alpha_1} \wedge \dots \wedge \vartheta^{\alpha_n} = *\mathbf{1}, \quad (104)$$

dual to the unit zero-form. Picking a (pseudo-)orthonormal positively oriented coframe $\overset{\circ}{\vartheta}^\alpha$, the g -volume element simplifies to

$$\eta = \overset{\circ}{\vartheta}^{\hat{1}} \wedge \dots \wedge \overset{\circ}{\vartheta}^{\hat{n}}. \quad (105)$$

Having this n -form at our disposal, we can successively contract it by means of the frame e_α , thereby arriving at an $(n-1)$ -form, an $(n-2)$ -form, etc., until we terminate the series with a zero-form:

$$\begin{aligned} \eta_{\alpha_1} &:= e_{\alpha_1} \rfloor \eta = \frac{1}{(n-1)!} \eta_{\alpha_1\alpha_2\dots\alpha_n} \vartheta^{\alpha_2} \wedge \dots \wedge \vartheta^{\alpha_n} = *\vartheta_{\alpha_1}, \\ \eta_{\alpha_1\alpha_2} &:= e_{\alpha_2} \rfloor \eta_{\alpha_1} = \frac{1}{(n-2)!} \eta_{\alpha_1\alpha_2\alpha_3\dots\alpha_n} \vartheta^{\alpha_3} \wedge \dots \wedge \vartheta^{\alpha_n} = *(\vartheta_{\alpha_1} \wedge \vartheta_{\alpha_2}), \\ &\vdots \\ \eta_{\alpha_1\dots\alpha_n} &:= e_{\alpha_n} \rfloor \dots \rfloor e_{\alpha_1} \rfloor \eta = *(\vartheta_{\alpha_1} \wedge \dots \wedge \vartheta_{\alpha_n}). \end{aligned} \quad (106)$$

Thus, in four dimensions, which we are concentrating on, we have an η -basis at our disposal $(\eta, \eta_\alpha, \eta_{\alpha\beta}, \eta_{\alpha\beta\gamma}, \eta_{\alpha\beta\gamma\delta})$ that we will meet in numerous applications.

7. Field equations of metric–affine gauge theory (MAG)

The gauge procedure of the affine group led to the identification of the gauge potentials of the MAG, see Table 5. The material currents are then expected to couple to these potentials in the Yang-Mills type fashion, as is indicated in Fig. 1, and as we know already from the material energy-momentum current and the

coframe according to the line after (77). If Ψ , as a p -form, represents a matter field (fundamentally a representation of the $SL(4, R)$ or of some of its subgroups^{61,62}), its first order Lagrangian L will be embedded in metric-affine spacetime by the minimal coupling procedure, that is, exterior covariant derivatives feature in the kinetic terms of the Lagrangian instead of only exterior ones. Then the material currents are defined as follows:

$$\sigma^{\alpha\beta} := 2 \frac{\delta L}{\delta g_{\alpha\beta}}, \quad \Sigma_\alpha := \frac{\delta L}{\delta \vartheta^\alpha}, \quad \Delta^\alpha{}_\beta := \frac{\delta L}{\delta \Gamma^\alpha{}_\beta}. \quad (107)$$

From GR we expect that $\sigma^{\alpha\beta}$ is the metric (and symmetric) energy-momentum current of matter ('Hilbert current'), whereas Secs. 4 and 5 lead us to believe that Σ_α should be the corresponding canonical energy-momentum current ('Noether current'). The canonical hypermomentum current $\Delta^\alpha{}_\beta$, which couples to the linear connection, can be decomposed according to

$$\begin{aligned} \Delta_{\alpha\beta} &= \tau_{\alpha\beta} + \frac{1}{4} g_{\alpha\beta} \Delta + \widehat{\mathcal{X}}_{\alpha\beta} \\ &\sim \text{spin current} \oplus \text{dilation current} \oplus \text{shear current}, \end{aligned} \quad (108)$$

with $\tau_{\alpha\beta} := \Delta_{[\alpha\beta]}$ as (dynamical) spin current, $\Delta := \Delta^\gamma{}_\gamma$, as dilation current, and $\widehat{\mathcal{X}}_{\alpha\beta}$ as symmetric and tracefree shear current.

If one applies the Noether theorem to the material Lagrangian, then, Σ_α , $\tau_{\alpha\beta}$, and Δ can be identified with the special-relativistic canonical Noether currents of energy-momentum, spin, and dilation, respectively. This leaves only the shear current $\widehat{\mathcal{X}}_{\alpha\beta}$, see our earlier work²¹ for a detailed discussion, as a concept that is a bit more remote from direct observation than the other currents. In any case, the definitions (107) and their physical interpretations are completely justified by these considerations.

Adding to the matter Lagrangian a metric-affine first order gauge Lagrangian V ,

$$W = \int \left[V(g_{\alpha\beta}, \vartheta^\alpha, Q_{\alpha\beta}, T^\alpha, R_\alpha{}^\beta) + L(g_{\alpha\beta}, \vartheta^\alpha, \Psi, D\Psi) \right], \quad (109)$$

and applying the action principle, we find²⁹ the matter and the gauge field equations (of Yang-Mills type):

$$D \left(\frac{\partial L}{\partial (D\Psi)} \right) - (-1)^p \frac{\partial L}{\partial \Psi} = 0, \quad (\text{MATTER}) \quad (110)$$

$$D \left(2 \frac{\partial V}{\partial Q_{\alpha\beta}} \right) + 2 \frac{\partial V}{\partial g_{\alpha\beta}} = -\sigma^{\alpha\beta}, \quad (\text{ZEROTH}) \quad (111)$$

$$D \left(\frac{\partial V}{\partial T^\alpha} \right) + \frac{\partial V}{\partial \vartheta^\alpha} = -\Sigma_\alpha, \quad (\text{FIRST}) \quad (112)$$

$$D \left(\frac{\partial V}{\partial R_\alpha{}^\beta} \right) + \vartheta^\alpha \wedge \frac{\partial V}{\partial T^\beta} + 2 g_{\beta\gamma} \frac{\partial V}{\partial Q_{\alpha\gamma}} = -\Delta^\alpha{}_\beta. \quad (\text{SECOND}) \quad (113)$$

In SECOND a Noether identity for V has already been employed. Analogously, in FIRST, the canonical energy-momentum of the translational gauge potential ϑ^α can be expressed explicitly as

$$\frac{\partial V}{\partial \vartheta^\alpha} = e_\alpha \lrcorner V - (e_\alpha \lrcorner T^\beta) \wedge \frac{\partial V}{\partial T^\beta} - (e_\alpha \lrcorner R_{\beta\gamma}) \wedge \frac{\partial V}{\partial R_{\beta\gamma}} - (e_\alpha \lrcorner Q_{\beta\gamma}) \frac{\partial V}{\partial Q_{\beta\gamma}}. \quad (114)$$

This structure is known from Minkowski's energy-momentum tensor of the Maxwellian field. It is interesting to note that, provided SECOND is fulfilled, FIRST and ZEROTH are equivalent, i.e., one of them is redundant. It is for this reason that we abstain, for $\partial V/\partial g_{\alpha\beta}$, to display a formula similar to (114).

The field equation of a Yang-Mills theory reads $\overset{A}{D} (\partial V_{\text{YM}}/\partial F) = -J$. The field equations ZEROTH, FIRST, and SECOND are of this type. However, because of the universality of 'external' interactions (gravitation), additional *tensor-valued gauge currents* ($\partial V/\partial \vartheta^\alpha$ etc.) surface in the field equations. This is the distinguishing feature of MAG as compared to gauge theories of internal groups ($U(1)$, $SU(2)$, ...). In Yang-Mills theory the non-linearity of the gauge field is hidden in the non-tensorial pieces of the gauge covariant exterior derivative $\overset{A}{D}$ occurring on the left hand side $\overset{A}{D} (\partial V_{\text{YM}}/\partial F)$ of the Yang-Mills equation. In gravitational gauge theory, there are *additional* non-linearities, besides those in $\overset{\Gamma}{D}$, namely the ones represented by (114) etc.. This result is non-trivial as a simple argument will show:

Suppose we had a Hilbert-Einstein gauge Lagrangian linear in the curvature $R_\alpha{}^\beta$. Then in FIRST the leading term in differential order on its left hand side will vanish and we are left with (114). The surviving terms are

$$\frac{\partial V_{\text{HE}}}{\partial \vartheta^\alpha} \sim e_\alpha \lrcorner V_{\text{HE}} - (e_\alpha \lrcorner R_{\beta\gamma}) \wedge \frac{\partial V_{\text{HE}}}{\partial R_{\beta\gamma}} \sim -\Sigma_\alpha. \quad (115)$$

Since $\partial V_{\text{HE}}/\partial R_{\beta\gamma}$ will be a constant, we recover the Einstein three-form (corresponding to the Einstein tensor in Ricci calculus) from this equation, giving substance to Schrödinger's dictum⁸³ that the left hand side of Einstein's equation is, in some sense, the gravitational energy-momentum tensor. Consequently, Einstein's field equation of GR is encapsulated in $\partial V/\partial \vartheta^\alpha$ of FIRST and, as such, has a distinctive anti-Yang-Mills flavor. In contrast, the Einsteinian teleparallelism GR_{\parallel} , with its torsion-square Lagrangian V_{\parallel} , picks up an essential piece from the proper Yang-Mills term of FIRST,

$$D \frac{\partial V_{\parallel}}{\partial T^\alpha} + e_\alpha \lrcorner V_{\parallel} - (e_\alpha \lrcorner T^\beta) \wedge \frac{\partial V_{\parallel}}{\partial T^\beta} \sim -\Sigma_\alpha, \quad (116)$$

compare (77). It has much more of the Yang-Mills spirit than GR has – and this is the reason why GR_{\parallel} turned up when we gauged the translations in Secs. 4 and 5.

If, within our formalism, one desires to correctly derive the field equation for GR, see (115), and for GR_{\parallel} , see (116), then one has to put on *Lagrange multipliers*. In the

case of (115) they have to kill nonmetricity and torsion and for (116) to remove nonmetricity and curvature. The details have been worked out in our earlier review²⁹, see also Kopczyński³⁸. **MAG** (metric-affine gauge theory, the general framework), **PG** (Poincaré gauge theory, vanishing nonmetricity, hence in a Riemann-Cartan spacetime), **EC** (Einstein-Cartan theory, the PG with the curvature scalar as gravitational Lagrangian), **GR_{||}** (Einsteinian teleparallelism, vanishing nonmetricity, vanishing curvature, hence in a Weitzenböck spacetime, specific torsion square Lagrangian), and **GR** (general relativity, vanishing nonmetricity, vanishing torsion, hence in a Riemannian space, curvature scalar as Lagrangian) are different (sub-)cases of this general scheme.

A further remark is in order: Gauging the affine group yields the gauge potentials $(\vartheta^\alpha, \Gamma_\alpha^\beta)$, see Table 5. If a metric exists on top of that linearly connected manifold, then a further independent geometrical field variable is at hand. Following Trautman⁸⁸, we are taking $(g_{\alpha\beta}, \vartheta^\alpha, \Gamma_\alpha^\beta)$ as independent variables in the action (109). Because of the redundancy of ZEROTH or FIRST, *provided* SECOND is fulfilled, one could argue that one should drop, say, the coframe as independent variable, as done by Tucker^f and Wang⁹⁵, for example. In the earlier metric-affine unified field theories à la Einstein¹³(App.2) and Schrödinger⁸³, the coframe didn't even show up since they worked in a holonomic formalism (unsuitable for representing fermions). Because of various arguments, however, we feel more comfortable with our procedure: (i) Both energy-momentum currents, the Hilbertian $\sigma^{\alpha\beta}$ and the Noetherian Σ_α , have a good and direct physical interpretation in SR. Dropping a gauge variable means dropping one of these useful quantities as a fundamental current. (ii) *A posteriori*, we do find the identity causing the redundancy mentioned. This seems safer than to assume something to that extend *a priori*. (iii) If we dropped the metric as an independent variable, for example, then we would have to take *orthonormal* frames as field variables, giving away a piece of freedom which we had earlier in being able to work with whatever frame we liked, be it orthonormal or oblique.

Let us finally remind ourselves: As soon as the gauge Lagrangian V is specified explicitly, we can find the field equations ZEROTH, FIRST, and SECOND by sheer partial differentiation. Hence the using of the general form of the field equations may save a lot of work.

8. Model building: Einstein-Cartan theory and beyond

The missing piece within the framework that we finally established in Sec. 7 is the gauge field Lagrangian $V(g_{\alpha\beta}, \vartheta^\alpha, Q_{\alpha\beta}, T^\alpha, R_\alpha^\beta)$. The hope is that the model, with a suitably chosen V – perhaps combined with some symmetry breaking mechanism which, for example, reduces the linear group $GL(4, R)$ to the Lorentz group $SO(1, 3)$ – can be consistently quantized. For 1+1 dimensional curvature square models^{51,64,67}

^fWe are grateful to Robin W. Tucker (Lancaster) for an interesting discussion on this question.

successful quantization methods are already available, see Kloesch and Strobl³⁷ and the literature given there.

In testing a new framework, one first wants, in some limit, to recover old ground where one feels at home. In our case this is GR. There are at least two ways of how to achieve this: One can choose the curvature scalar à la Hilbert (in exterior calculus: the corresponding four-form $R^{\alpha\beta} \wedge \eta_{\alpha\beta}$ à la Trautman⁸⁸) and take Lagrange multipliers for extinguishing nonmetricity and torsion:

$$V_{\text{GR}} = -\frac{1}{2\ell^2} R^{\alpha\beta} \wedge \eta_{\alpha\beta} + \frac{1}{2} Q_{\alpha\beta} \wedge {}^{(1)}\lambda^{\alpha\beta} + T^\alpha \wedge {}^{(2)}\lambda_\alpha. \quad (117)$$

Note that $R_\alpha{}^\beta$ is the curvature tensor of the independent field variable $\Gamma_\alpha{}^\beta$, for the η 's see (106).

8.1. Einstein-Cartan theory EC

A second way is to start with the Einstein-Cartan Lagrangian (here with cosmological constant)

$$V_{\text{EC}} = -\frac{1}{2\ell^2} R^{\alpha\beta} \wedge \eta_{\alpha\beta} + \frac{1}{2} \Lambda \eta + \frac{1}{2} Q_{\alpha\beta} \wedge \lambda^{\alpha\beta}, \quad (118)$$

and to derive the corresponding field equations:

$$\frac{1}{2} \eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma} + \Lambda \eta_\alpha = \ell^2 \Sigma_\alpha, \quad (119)$$

$$\frac{1}{2} \eta_{\alpha\beta\gamma} \wedge T^\gamma = \ell^2 \tau_{\alpha\beta}. \quad (120)$$

In components in a holonomic frame they read:

$$Ric_{ij} - \frac{1}{2} g_{ij} Ric^k{}_k + \Lambda g_{ij} = \ell^2 \Sigma_{ij}, \quad (121)$$

$$T_{ij}{}^k + 2 \delta_{[i}^k T_{j]l}{}^l = \ell^2 \tau_{ij}{}^k. \quad (122)$$

We recover GR for vanishing spin $\tau_{\alpha\beta} = 0$. In this context only the vanishing of nonmetricity had to be assumed. The vanishing of torsion, for spinless matter, was the result of the second field equation (120). We will see below, in Sec. 8.2, that also the last Lagrange multiplier can be abandoned if one amends the gravitational part of the Lagrangian with a piece quadratic in the Weyl one-form.

Basically, the two EC field equations (119) and (120) are *first order* partial differential equations in $\Gamma_\alpha{}^\beta$ and ϑ^α , with a spin fluid as source, see ref.⁽⁶⁵⁾. As we explained already in the last Sec. 7, this is the anti-Yang-Mills flavor of GR or EC caused by the absence of an explicit torsion piece in the Lagrangian. A physical consequence is that in EC we have the usual gravitational interaction of the Newton-Einstein type

plus a very weak (non-propagating) spin contact interaction. Up to the fifties, *weak* interaction was also thought to be of contact type (in fact, of a vector-axial vector type). Later, following gauge ideas, the short-range intermediate W -boson (and the Z) were postulated that made the weak interaction propagate.

8.2. Poincaré gauge theory PG, the quadratic version

The simplest appearance of explicit torsion pieces is that in the $\text{GR}_{||}$ Lagrangian^{24,25} studied in Sec. 5:

$$V_{||} = -\frac{1}{2\ell^2} T^\alpha \wedge * \left(-{}^{(1)}T_\alpha + 2{}^{(2)}T_\alpha + \frac{1}{2}{}^{(3)}T_\alpha \right) + \frac{1}{2} Q_{\alpha\beta} \wedge {}^{(1)}\lambda^{\alpha\beta} + R_\alpha{}^\beta \wedge {}^{(2)}\lambda^\alpha{}_\beta. \quad (123)$$

This can be considered as a starting point for turning to Lagrangians quadratic in the field strengths. Amongst the simplest model cases is the *purely* quadratic von der Heyde et al.^{32,6} Lagrangian

$$V_{\text{vdH}} = -\frac{1}{2\ell^2} (T^\alpha \wedge \vartheta^\beta) \wedge * (T_\beta \wedge \vartheta_\alpha) - \frac{1}{2\kappa} R^{\alpha\beta} \wedge * R_{\alpha\beta} + \frac{1}{2} Q_{\alpha\beta} \wedge \lambda^{\alpha\beta}. \quad (124)$$

This Lagrangian has been ‘derived’ by means of the Gordon decomposition argument^{32,25}, see also Rumpf⁸¹. It may have problems with the positivity of the energy, see refs.^(85,41,97), but the situation is not completely clear to us. The torsion square piece in this Lagrangian, in a Weitzenböck spacetime, has a classical Newtonian limit. It differs from the torsion pieces in the teleparallel Lagrangian (123) by a quadratic axial torsion piece (then in (123) we had $-{}^{(3)}T_\alpha$ instead).

A number of exact classical solutions has been found for the model (124), a Kerr-NUT solution with torsion⁴ is amongst the most prominent ones. For illustrating the basic features of such solutions, we display the less complicated subcase, namely the Baekler-Lee^{3,43} solution – this is the Reissner-Nordström solution with dynamic torsion – as a fairly transparent example. We choose Schwarzschild coordinates (t, r, θ, ϕ) , $M = \text{mass}$, $q = \text{electric charge}$, and find the orthonormal coframe

$$\begin{aligned} \vartheta^{\hat{t}} &= \frac{1}{2} [(\Phi + 1)dt + (1 - \frac{1}{\Phi})dr], \\ \vartheta^{\hat{r}} &= \frac{1}{2} [(\Phi - 1)dt + (1 + \frac{1}{\Phi})dr], \\ \vartheta^{\hat{\theta}} &= r d\theta, \\ \vartheta^{\hat{\phi}} &= r \sin \theta d\phi. \end{aligned} \quad (125)$$

The corresponding metric reads:

$$ds^2 = -\Phi dt^2 + \frac{1}{\Phi} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (126)$$

In the Reissner-Nordström function

$$\Phi := 1 - \frac{2(Mr - q^2)}{r^2} - \frac{\kappa}{4\ell^2}r^2 \quad (127)$$

the ‘cosmological term’ is induced by the Yang-Mills type curvature square piece in the Lagrangian (‘strong gravity’, cf. ref.(⁶³)). If we additionally had a ‘naked’ cosmological constant, we could shift its value by a suitable choice of the ‘strong’ coupling constant κ . Torsion and curvature are given by

$$\begin{aligned} T^{\hat{t}} = T^{\hat{r}} &= \frac{Mr - 2q^2}{r^3} \vartheta^{\hat{t}} \wedge \vartheta^{\hat{r}}, \\ T^{\hat{\theta}} &= \frac{Mr - q^2}{r^3} \left(\vartheta^{\hat{t}} \wedge \vartheta^{\hat{\theta}} - \vartheta^{\hat{r}} \wedge \vartheta^{\hat{\theta}} \right), \\ T^{\hat{\varphi}} &= \frac{Mr - q^2}{r^3} \left(\vartheta^{\hat{t}} \wedge \vartheta^{\hat{\varphi}} - \vartheta^{\hat{r}} \wedge \vartheta^{\hat{\varphi}} \right), \end{aligned} \quad (128)$$

and

$$R^{\alpha\beta} = \frac{\kappa}{4\ell^2} \vartheta^\alpha \wedge \vartheta^\beta + \frac{Mr - q^2}{r^2} {}^{(4)}\mathcal{R}^{\alpha\beta}, \quad (129)$$

respectively, where

$$\begin{aligned} {}^{(4)}\mathcal{R}^{\hat{t}\hat{\theta}} &= {}^{(4)}\mathcal{R}^{\hat{r}\hat{\theta}} := \frac{\kappa}{4\ell^2} \left(\vartheta^{\hat{t}} \wedge \vartheta^{\hat{\theta}} - \vartheta^{\hat{r}} \wedge \vartheta^{\hat{\theta}} \right), \\ {}^{(4)}\mathcal{R}^{\hat{t}\hat{\varphi}} &= {}^{(4)}\mathcal{R}^{\hat{r}\hat{\varphi}} := \frac{\kappa}{4\ell^2} \left(\vartheta^{\hat{t}} \wedge \vartheta^{\hat{\varphi}} - \vartheta^{\hat{r}} \wedge \vartheta^{\hat{\varphi}} \right), \end{aligned} \quad (130)$$

represent a tracefree symmetric Ricci piece of the curvature two-form. The Coulomb field shows up in the electromagnetic field strength:

$$F = \frac{2q}{\ell r^2} \vartheta^{\hat{t}} \wedge \vartheta^{\hat{r}}. \quad (131)$$

It is a spherically symmetric vacuum solution with Maxwell field, i.e. a Reissner-Nordström solution with dynamic torsion. We don’t display the solution in its original frame but in a suitably rotated one⁵ such that the torsion two-form (128) has a ‘Coulombic’ look without global extra factors in front of the corresponding expressions. Note the factor of *two* in the $T^{\hat{t}}$ component of (128) in the q^2 -piece. One interesting feature of this solution is that the curvature square Lagrangian supplies a constant curvature background proportional to κ , see the first piece on the right hand side of (129). The *torsion* is – this is not an unexpected feature in the light of our teleparallelism ‘philosophy’ – induced by ordinary *Newton-Einstein gravity*, as can be read off from (128), a fact usually hard to swallow by colleagues who relate torsion with obscurity. In (128) torsion is visibly the translation field strength, a fact made possible by the purely quadratic torsion piece in the Lagrangian, without a Hilbert-Einstein type admixture.

The von der Heyde Lagrangian (124) is a subcase of the general quadratic PG Lagrangian^{25,17,66}

$$V_{\text{QPG}} = \frac{\Lambda}{\ell^2} \eta + \frac{a_0}{2\ell^2} R^{\alpha\beta} \wedge \eta_{\alpha\beta} + \frac{1}{2\ell^2} T^\alpha \wedge * \left(\sum_{M=1}^3 a_{(M)} {}^{(M)}T_\alpha \right) + \frac{1}{2\kappa} R^{\alpha\beta} \wedge * \left(\sum_{N=1}^6 b_{(N)} {}^{(N)}R_{\alpha\beta} \right) + \frac{1}{2} Q_{\alpha\beta} \wedge \lambda^{\alpha\beta}. \quad (132)$$

Each of the three irreducible torsion and six irreducible curvature pieces contributes to the Lagrangian with an individual weight. The propagating modes of this Lagrangian were investigated by Sezgin and van Nieuwenhuizen⁸⁵ and by Kuhfuss and Nitsch⁴¹. A subclass of Lagrangians survived their selection criteria motivated by quantum field theoretical considerations (ghost-freeness, positive energy). Minkevich^{54,55}, amongst others, studied Friedmann type cosmological models resulting from such a Lagrangian.

8.3. Coupling to a scalar field

It is near at hand to add a dilaton type massless scalar field to this Lagrangian:

$$V_{\Phi\text{grav}} = V_{\text{QPG}} + \frac{1}{2} d\Phi \wedge *d\Phi. \quad (133)$$

We found for this model a remarkable exact solution, a *torsion kink*⁶. Attributing distinctive Higgs-like features to the scalar field, we arrive at the more general model of Floreanini and Percacci¹⁶:

$$V_{\text{FP}} = c_1 d\Phi \wedge *d\Phi + c_2 d\Phi \wedge *(e_\mu \rfloor T^\mu) + U(\Phi) + \frac{a_0}{2\ell^2} \Phi^2 R^{\alpha\beta} \wedge \eta_{\alpha\beta} + \frac{1}{2\ell^2} T^\alpha \wedge * \left(\sum_{M=1}^3 a_M {}^{(M)}T_\alpha \right) + \frac{1}{2} R^{\alpha\beta} \wedge * \left(\sum_{N=1}^6 b_N {}^{(N)}R_{\alpha\beta} \right) + \frac{1}{2} Q_{\alpha\beta} \wedge \lambda^{\alpha\beta}. \quad (134)$$

The scalar Φ couples to the EC term in a Jordan-Brans-Dicke type way. The explicit form of the ‘Higgs’ potential $U(\Phi)$ is left open. Note the direct coupling $d\Phi \wedge *T$ in (134) which is, however, odd in Φ . The authors of (134), on a quantum field theoretical level, investigated the renormalizability properties of their model.

Being relativists, we would be ill-advised if we didn’t try to give the scalar field Φ a geometrical meaning, perhaps in the context of the Weyl one-form Q which is of the type of a gauge potential for dilations anyways. Therefore we lift the last Lagrange multiplier and now turn to...

8.4. Metric-affine gauge theory MAG

Let us, however, first continue the discussion of above of how to arrive at GR in spite of relaxing the last constraint and liberating thereby the connection completely from its dominance by the metric. The naive way, namely just to take a term

proportional to $R^{\alpha\beta} \wedge \eta_{\alpha\beta}$, doesn't work. The projective transformation

$$\Gamma_{\alpha}{}^{\beta} \longrightarrow \Gamma_{\alpha}{}^{\beta} + \delta_{\alpha}^{\beta} P, \quad (135)$$

with some one-form field P , leaves the Hilbert-Einstein type Lagrangian invariant. Consequently, in such a model, the connection would only be determined up to a one-form (with four components). This is unsatisfactory. Moreover, if one coupled the gauge fields to matter, then only projectively invariant matter Lagrangians would be allowed, an a priori constraint without physical justification. Therefore one has to remove the projective invariance from the gravitational Lagrangian.

Since a projective transformation changes the trace $\Gamma_{\gamma}{}^{\gamma}$ of a connection and this trace is closely related to the Weyl one-form,

$$\Gamma_{\gamma}{}^{\gamma} = 2Q + d \ln \sqrt{|\det g_{\alpha\beta}|}, \quad d\Gamma_{\gamma}{}^{\gamma} = R_{\gamma}{}^{\gamma} = 2dQ, \quad (136)$$

see Eq.(3.10.13) of ref.(29), an obvious way to remove the projective invariance is to add the square of the Weyl one-form^{26,47}:

$$V_{\text{GR}'} = -\frac{1}{2\ell^2} \left(R^{\alpha\beta} \wedge \eta_{\alpha\beta} + \beta Q \wedge *Q \right). \quad (137)$$

A moment's reflection will remind ourselves what we have achieved by this innocently looking Lagrangian (137): Varying metric, frame, and connection independently, in vacuum – that is, in the absence of matter – yields the *Einstein vacuum equation in a Riemannian spacetime*. Generically, if matter is present and supplies energy-momentum and hypermomentum currents, then the *hypermomentum*, via the second field equation (113), turns out to be proportional to the *post-Riemannian* pieces of the connection.

The problem with the Lagrangian (137) is that it has the same defect as the EC Lagrangian. We have Newton-Einstein gravity and no further propagating modes. The obvious remedy is to add kinetic terms and, since we called the Weyl one-form already in earlier, we may want to make these modes propagating^{28,78}:

$$V_{\text{GRQ}} = -\frac{1}{2\ell^2} \left(R^{\alpha\beta} \wedge \eta_{\alpha\beta} + \beta Q \wedge *Q \right) - \frac{\alpha}{2} dQ \wedge *dQ. \quad (138)$$

In our first order formalism the Lagrangian must be expressed in terms of the gauge potential and their first derivatives. Since Q is already on the level of a field strength, one may be tempted to forbid the kinetic Q -terms. However, by means of the trace of the zeroth Bianchi identity, $dQ = \frac{1}{2}R_{\gamma}{}^{\gamma}$, we can cast out the pseudo-second order terms and find the well-behaved first order Lagrangian

$$V_{\text{GRQ}} = -\frac{1}{2\ell^2} \left(R^{\alpha\beta} \wedge \eta_{\alpha\beta} + \beta Q \wedge *Q \right) - \frac{\alpha}{8} R_{\alpha}{}^{\alpha} \wedge *R_{\beta}{}^{\beta}. \quad (139)$$

This might be the *simplest* reasonable metric-affine Lagrangian with propagating dilation modes. Probably it deserves closer investigation. If one amended (139) with

a piece $\sim R^{\alpha\beta} \wedge {}^*(3)Z_{\alpha\beta}$, see (140), then one would expect the simplest shear modes to arise. Here $Z_{\alpha\beta} := R_{(\alpha\beta)}$ is the symmetric (post-Riemannian) and, for later use, $W_{\alpha\beta} := R_{[\alpha\beta]}$ the antisymmetric piece of the curvature two-form.

The trace $R_{\beta}{}^{\beta}$ of the curvature in (139) represents an irreducible piece of the curvature, in fact the piece which we numbered as *ten* and called, in our computer algebra programs⁸⁴, DILCURV. Altogether, in a metric-affine space, the curvature has *eleven* irreducible pieces, see ref.⁽²⁹⁾, Table 4. If we recall that the nonmetricity has *four* irreducible pieces, then the general quadratic Lagrangian in MAG reads:

$$\begin{aligned} V_{\text{QMA}} = & \frac{1}{2\ell^2} \left[-a_0 R^{\alpha\beta} \wedge \eta_{\alpha\beta} + 2\Lambda \eta + T^\alpha \wedge {}^* \left(\sum_{I=1}^3 a_I {}^{(I)}T_\alpha \right) \right. \\ & + 2 \left(\sum_{I=2}^4 c_I {}^{(I)}Q_{\alpha\beta} \right) \wedge \vartheta^\alpha \wedge {}^*T^\beta + Q_{\alpha\beta} \wedge {}^* \left(\sum_{I=1}^4 b_I {}^{(I)}Q^{\alpha\beta} \right) \left. \right] \\ & + \frac{1}{2} R^{\alpha\beta} \wedge {}^* \left(\sum_{I=1}^6 w_I {}^{(I)}W_{\alpha\beta} + \sum_{I=1}^5 z_I {}^{(I)}Z_{\alpha\beta} \right). \end{aligned} \quad (140)$$

In spite of this affluence of generality, Tresguerres^{91,g} was able to find two exact solutions of the corresponding vacuum field equations in a most remarkable piece of work. He also discussed⁹¹ the reasons for introducing the mixed $Q \wedge \vartheta \wedge {}^*T$ term. Generically Tresguerres found a Baekler-Lee type solution with torsion, see (125) to (131) above, but the *dilaton charge takes the place of the electric charge*. Thus the Weyl one-form Q , the quasiMaxwellian potential, has a $1/r$ -behavior and is closely interwoven with the torsion vector. In the second solution, the metric is the same, but, additionally, two types of *shear charges* emerge. The Tresguerres solutions with dilation and shear charges are something qualitatively new in gauge models of gravity.

Tucker and Wang⁹⁵ put $\beta = 0$ in (138) or (139),

$$V_{\text{TW}} = -\frac{1}{2\ell^2} R^{\alpha\beta} \wedge \eta_{\alpha\beta} - \frac{\alpha}{2} dQ \wedge {}^*dQ = -\frac{1}{2\ell^2} R^{\alpha\beta} \wedge \eta_{\alpha\beta} - \frac{\alpha}{8} R_\alpha{}^\alpha \wedge {}^*R_\beta{}^\beta, \quad (141)$$

i.e. they excluded the massive term of the Weyl field from further consideration. Rather, they explored the analogy of the Weyl one-form Q to the Maxwell potential A under these circumstances. Their field equations can be read off from (112), (113), and (114) straightforwardly as,

$$-e_\alpha \rfloor V_{\text{TW}} + (e_\alpha \rfloor R_{\beta}{}^\gamma) \wedge \frac{\partial V_{\text{TW}}}{\partial R_{\beta}{}^\gamma} = \Sigma_\alpha, \quad (142)$$

$$-D \left(\frac{\partial V_{\text{TW}}}{\partial R_\alpha{}^\beta} \right) = \Delta^\alpha{}_\beta, \quad (143)$$

or

$$\frac{1}{2} \eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma} + \frac{\alpha}{8} \left[(e_\alpha \rfloor R_{\beta}{}^\beta) \wedge {}^*R_\gamma{}^\gamma - (e_\alpha \rfloor {}^*R_{\beta}{}^\beta) \wedge R_\gamma{}^\gamma \right] = \ell^2 \Sigma_\alpha, \quad (144)$$

⁹See also Tresguerres^{89,90,92,93}, Macías et al.⁴⁴, and, for cosmological models, Minkevich⁵⁶.

$$\frac{1}{2} D\eta^\alpha{}_\beta - \frac{\alpha}{4} \delta_\beta^\alpha d^*R_\gamma{}^\gamma = \ell^2 \Delta^\alpha{}_\beta. \quad (145)$$

Tucker and Wang⁹⁵ found Baekler-Lee type vacuum solutions with dilation (‘Weyl’) charge, just as Tresguerres, but, in addition, they presented a highly interesting solution with propagating massless spinor matter as source.

Nevertheless, one should recognize that the Lagrangian (141) is not without problems. Because the massive piece $\sim Q \wedge *Q$ is missing, the gauge Lagrangian V_{TW} is invariant under the special projective transformation

$$\Gamma_\alpha{}^\beta \longrightarrow \Gamma_\alpha{}^\beta + \delta_\alpha^\beta dp. \quad (146)$$

Whereas such a type of ‘gauge’ transformation is desirable for an internal $U(1)$ -connection – like in Maxwell’s theory – it is definitely dangerous in the context of a dilation transformation. The linear connection is only determined up to the transformation (146), that is, not all of the 64 components are uniquely determined in the TW-model, see also the analysis of Teyssandier and Tucker⁸⁶.

The case studies of Tresguerres and Tucker-Wang taught us that the Weyl one-form plays a particular role in representing a dilation type field that ought to be useful in the context of the breaking of the dilation symmetry. Therefore there were attempts by Mielke et al., see ref.⁽²⁹⁾ Sec. 6, to superimpose on the Lagrangian (140) an additional conformal symmetry in order to have a massless theory, free of dimensionful coupling constants at the beginning. Then one couples to a hypothetical dilaton field σ ,

$$V = -\frac{\sigma^2}{2} R^{\alpha\beta} \wedge \eta_{\alpha\beta} + \frac{1}{2} (D\sigma) \wedge *D\sigma + \frac{\lambda}{4} \sigma^4 \eta, \quad (147)$$

and breaks the dilation symmetry, thereby arriving at a low energy massive Lagrangian. Analogous approaches were proposed by Gregorash and Papini^{19,20}, Hochberg and Plunien³³, and by Poberii⁷⁷. Recently Pawłowski and Rączka^{75,74,94} developed similar models in a Riemannian spacetime, though. Nevertheless, they required conformal invariance for the Lagrangian they started from, coupled to a dilation field etc.. There are close relationships between the gravitational sectors of these models which lead us to the belief that the P&R model should be redrafted in the framework of a Weyl-Cartan, if not of a metric-affine spacetime.

With all these developments from different quarters, we have here – perhaps for the first time – a consistent framework for gauge models carrying both, non-trivial torsion *and* nonmetricity.

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10. References

1. A. Ali, J. Ellis, and S. Randjbar-Daemi (eds.): *Salamfestschrift*. A Collection of Talks from the Conference on Highlights of Particle and Condensed Matter Physics (World Scientific, Singapore, 1994).
2. J. Audretsch, F.W. Hehl, and C. Lämmerzahl: *Matter wave interferometry and why quantum objects are fundamental for establishing a gravitational theory*. In: ‘Relativistic Gravity Research’ Proc., Bad Honnef, Germany 1991, J. Ehlers and G. Schäfer, eds., Lecture Notes in Physics (Springer) **410** (1992) 368–407.
3. P. Baekler: *A spherically symmetric vacuum solution of the quadratic Poincaré gauge field theory of gravitation with Newtonian and confinement potentials*. Phys. Lett. **B99** (1981) 329–332.
4. P. Baekler, M. Gürses, F.W. Hehl, and J.D. McCrea: *The exterior gravitational field of a charged spinning source in the Poincaré gauge theory: A Kerr-Newman metric with dynamic torsion*. Phys. Lett. **A128** (1988) 245–250.
5. P. Baekler, R. Hecht, F.W. Hehl, and T. Shirafuji: *Mass and spin of exact solutions of the Poincaré gauge theory*. Progr. Theor. Phys. **78** (1987) 16–21.
6. P. Baekler, E.W. Mielke, R. Hecht, and F.W. Hehl: *Kinky torsion in a Poincaré gauge model of gravity coupled to a massless scalar field*. Nucl. Phys. **B288** (1987) 800–812.
7. J.D. Bjorken and S. Drell: *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).
8. U. Bonse and T. Wroblewski: *Measurement of neutron quantum interference*. Phys. Rev. Lett. **51** (1983) 1401–1404.
9. H.-H.v. Borzeszkowski and H.-J. Treder: *Mixed field theories and Weyl-Cartan geometry*. Lecture given at this Erice School (1995).
10. Y. M. Cho: *Einstein Lagrangian as the translational Yang-Mills Lagrangian*. Phys. Rev. **D14** (1976) 2521–2525.
11. R. Colella, A.W. Overhauser, and S.A. Werner: *Observation of Gravitationally Induced Quantum Interference*. Phys. Rev. Lett. **34** (1975) 1472–1474.
12. R. Debever (ed.): *Elie Cartan – Albert Einstein: Lettres sur le Parallélisme Absolu 1929–1932*. Original text with English translation. Académie Royale

- de Belgique and Princeton University Press 1979.
13. A. Einstein: *The Meaning of Relativity*, 5th ed. (Princeton University Press, Princeton, 1955).
 14. A. Einstein: *Preface as of 1955 April 04* in ‘Cinquant’anni di Relatività 1905–1955.’ M. Pantaleo, ed. Edizioni Giuntine and Sansoni Editore, Firenze (1955). Translation from the German by D. Hartley and the authors. The German original of the piece we quoted reads as follows: “...die wesentliche Leistung der allgemeinen Relativitätstheorie, nämlich die Überwindung (des) ‘starren’ Raumes, d.h. des Inertialsystems, ist *nur indirekt* mit der Einführung einer Riemann-Metrik verbunden. Das unmittelbar wesentliche begriffliche Element ist das die infinitesimale Verschiebung von Vektoren ausdrückende ‘Verschiebungsfeld’ (Γ^l_{ik}). Dieses nämlich ersetzt den durch das Inertialsystem gesetzten Parallelismus räumlich beliebig getrennter Vektoren (nämlich Gleichheit entsprechender Komponenten) durch eine infinitesimale Operation. Dadurch wird die Bildung von Tensoren durch Differentiation ermöglicht und so die Einführung des ‘starren’ Raumes (Inertialsystem) entbehrlich gemacht. Dem gegenüber erscheint es in gewissem Sinne von sekundärer Wichtigkeit, dass ein besonderes Γ -Feld sich aus der Existenz einer Riemann-Metrik deduzieren lässt...”
 15. R. Feynman: *Lectures on Gravitation*. Lecture notes by F.B. Morinigo and W.G. Wagner (California Institute of Technology, Pasadena, California 1962/63).
 16. R. Floreanini and R. Percacci: *Renormalization-group flow of a dilaton potential*. Phys. Rev. **D52** (1995) 896–911.
 17. H. Goenner: *Alternative theories of gravity*. In: ‘General Relativity and Gravitation, GR11’. M.A.H. MacCallum, ed. (Cambridge University Press, Cambridge, 1987) pp. 262–273.
 18. D.M. Greenberger and A.W. Overhauser: *The role of gravity in quantum theory*. Scientific American **242** (May 1980) 54–64.
 19. D. Gregorash and G. Papini: *Weyl–Dirac theory with torsion*. Nuovo Cimento **B55** (1980) 37–51.
 20. D. Gregorash and G. Papini: *Weyl–Dirac theory with torsion. II. Foundations and conservation equations*. Nuovo Cimento **B56** (1980) 21–38.
 21. F. Gronwald and F.W. Hehl: *Stress and hyperstress as fundamental concepts in continuum mechanics and in relativistic field theory*. In: ‘Advances in Modern Continuum Dynamics’, International Conference in Memory of Antonio Signorini, Isola d’Elba, June 1991. G. Ferrarese, ed. (Pitagora Editrice, Bologna, 1993) pp. 1–32.
 22. R.T. Hammond: *New fields in general relativity*. Contemporary Physics **36** (1995) 103–114.
 23. D. Hartley: *Normal frames for non-Riemannian connections*. Class. Quan-

- tum Grav. **12** (1995) L103–L105.
24. K. Hayashi and T. Shirafuji: *New general relativity*. Phys. Rev. **D19** (1979) 3524–3553.
 25. F.W. Hehl: *Four lectures on Poincaré gauge theory*. In: Proc. of the 6th Course of the School of Cosmology and Gravitation on ‘Spin, Torsion, Rotation, and Supergravity’, held at Erice, Italy, May 1979, P.G. Bergmann, V. de Sabbata, eds. (Plenum, New York, 1980) pp. 5–61.
 26. F.W. Hehl, G.D. Kerlick, and P. von der Heyde: *On a new metric affine theory of gravitation*. Phys. Lett. **63B** (1976) 446–448.
 27. F.W. Hehl, J. Lemke, and E.W. Mielke: *Two lectures on fermions and gravity*. In: ‘Geometry and Theoretical Physics’, J. Debrus and A.C. Hirshfeld (eds.) (Springer, Berlin, 1991) pp. 56–140.
 28. F.W. Hehl, E.A. Lord, L.L. Smalley: *Metric–affine variational principles in general relativity. II. Relaxation of the Riemannian constraint*. Gen. Relat. Grav. **13** (1981) 1037–1056.
 29. F.W. Hehl, J.D. McCrea, E.W. Mielke, and Y. Ne’eman: *Metric-Affine Gauge Theory of Gravity: Field Equations, Noether Identities, World Spinors, and Breaking of Dilation Invariance*. Phys. Rep. **258** (1995) 1–171.
 30. F.W. Hehl and W.-T. Ni: *Inertial effects of a Dirac particle*. Phys. Rev. **D42** (1990) 2045–2048.
 31. P. von der Heyde: *The equivalence principle in the U_4 theory of gravitation*. Lett. al Nuovo Cimento **14** (1975) 250–252.
 32. P. von der Heyde: *Is gravitation mediated by the torsion of spacetime?* Z. Naturf. **31a** (1976) 1725–1726.
 33. D. Hochberg and G. Plunien: *Theory of matter in Weyl spacetime*. Phys. Rev. **D43** (1991) 3358–3367.
 34. J.C. Huang: *Dirac particle, gravity, and inertial effects*. Ann. Physik (Leipzig) **3** (1994) 53–67.
 35. B.Z. Iliev: *Special bases for derivations of tensor algebras. I. Cases in the neighborhood and at a point. II. Case along paths. III. Case along smooth maps with separable points of selfintersection*. Preprints E5-92-507, E5-92-508, E5-92-543 (Joint Institute for Nuclear Research, Dubna, 1992).
 36. R. d’Inverno: *Introducing Einstein’s Relativity*. (Clarendon Press, Oxford, 1992).
 37. T. Kloesch and T. Strobl: *Classical and quantum gravity in 1+1 dimensions, part I: A unifying approach; part II: The universal coverings*. Electronic archive Los Alamos, gr-qc/9508020 and gr-qc/9511081 (1995).
 38. W. Kopczyński: *The Palatini principle with constraints*. Bull. de l’Academie Polon. des Sci., Sér. math. astr. et phys. **23** (1975) 467–473.
 39. W. Kopczyński: *Problems with metric-teleparallel theories of gravitation*. J. Phys. **A15** (1982) 493–506.

40. W. Kopczyński and A. Trautman: *Spacetime and Gravitation*. Translated from the Polish (Wiley, Chichester and PWN – Polish Scientific Publishers, Warszawa, 1992).
41. R. Kuhfuss and J. Nitsch: *Propagating modes in gauge field theories of gravity*. Gen. Relat. Grav. **18** (1986) 1207–1227.
42. C. Lämmerzahl: *On the equivalence principle in quantum theory*. to appear in Gen. Rel. Grav. (1996).
43. C.H. Lee: *A spherically symmetric electro-vacuum solution of the Poincaré gauge field theory of gravitation*. Phys. Lett. **B130** (1983) 257–259.
44. A. Macías, E.W. Mielke, and H.A. Morales-Técotl: *Torsion and Weyl connector in metric-affine models of gravity*, J. Math. Phys. **36** (1995) 5868–5976.
45. A. Macías, E.W. Mielke, and H.A. Morales-Técotl: *Chiral fermions coupled to chiral gravity*. In: ‘Proc. of CAP 50 anniversary’, E. Lesard et al., eds. (Springer Verlag, Berlin) in print.
46. B. Mashhoon: *Neutron interferometry in a rotating frame of reference*. Phys. Rev. Lett. **61** (1988) 2639–2642.
47. J.D. McCrea, F.W. Hehl, and E.W. Mielke: *Mapping Noether identities into Bianchi identities in general relativistic theories of gravity and in the field theory of static lattice defects*. Int. J. Theor. Phys. **29** (1990) 1185–1206.
48. E.W. Mielke: *Geometrodynamics of Gauge Fields — On the geometry of Yang-Mills and gravitational gauge theories* (Akademie-Verlag, Berlin, 1987).
49. E.W. Mielke: *Positive-gravitational-energy proof from complex variables?* Phys. Rev. **D42** (1990) 3388–3394.
50. E.W. Mielke: *Ashtekar’s complex variables in general relativity and its teleparallelism equivalent*. Ann. Phys. (N.Y.) **219** (1992) 78–108.
51. E.W. Mielke, F. Gronwald, Yu.N. Obukhov, R. Tresguerres, and F.W. Hehl: *Towards complete integrability of two dimensional Poincaré gauge gravity*. Phys. Rev. **D48** (1993) 3648–3662.
52. E.W. Mielke, Yu.N. Obukhov, and F.W. Hehl: *Yang-Mills configurations from 3D Riemann-Cartan geometry*. Phys. Lett. **A192** (1994) 153–162.
53. R. Mills: *Gauge fields*. American J. Physics **57** (1989) 493–507.
54. A.V. Minkevich: *Generalized cosmological Friedmann equations without gravitational singularity*. Phys. Lett. **A80** (1980) 232–234.
55. A.V. Minkevich: *Generalized cosmological Friedman equation and de Sitter solution*. Phys. Lett. **A95** (1983) 422–424.
56. A.V. Minkevich and I.M. Nemenman: *On the influence of the gravitating vacuum on the dynamics of homogeneous isotropic models in gauge theories of gravity*, Class. Quantum Grav. **12** (1995) 1259–1265.
57. C.W. Misner, K.S. Thorne, and J.A. Wheeler: *Gravitation*. (Freeman, San Francisco, 1973).
58. L. Mistura: *Cartan connection and defects in Bravais lattices*. Int. J. Theor.

- Physics **29** (1990) 1207–1218.
59. G. Modanese and M. Toller: *Radial gauge in Poincaré gauge field theories*. J. Math. Phys. **31** (1990) 452–458.
 60. H.A. Morales-Técotl, A. Macías, and E.W. Mielke (1995): *Geometrical phases and translations*. In: ‘1st Mexican School on Gravitation and Mathematical Physics’, A. Macías et al., eds. (World Scientific, Singapore) in print.
 61. Y. Ne’eman: *Gravitational interaction of hadrons: Band-spinor representations of $GL(n, R)$* . Proc. Nat. Acad. Sci. (USA) **74** (1977) 4157–4159.
 62. Y. Ne’eman: *Spinor-type fields with linear, affine and general co-ordinate transformations*. Ann. Inst. Henri Poincaré **A28** (1978) 369–378.
 63. Y. Ne’eman and Dj. Šijački: *Chromogravity explains ‘strong gravity’*. In ref. [1] pp. 540–551.
 64. Yu.N. Obukhov: *Two-dimensional Poincaré gauge gravity with matter*. Phys. Rev. **D50** (1994) 5072–5086.
 65. Yu.N. Obukhov and V.A. Korotky: *The Weysenhoff fluid in Einstein-Cartan theory*. Class. Quant. Grav. **4** (1987) 1633–1657.
 66. Yu.N. Obukhov, V.N. Ponomarev, and V.V. Zhytnikov: *Quadratic Poincaré gauge theory of gravity: a comparison with the general relativity theory*. Gen. Relat. Grav. **21** (1989) 1107–1142.
 67. Yu.N. Obukhov, S.N. Solodukhin, and E.W. Mielke: *Coupling of lineal Poincaré gauge gravity to scalar fields*. Class. Quantum Grav. **11** (1994) 3069–3079.
 68. L. O’Raifeartaigh: *Hidden gauge symmetry*. Rep. Progr. Phys. **42** (1979) 159–223.
 69. L. O’Raifeartaigh: *The dawning of gauge theory*. In ref.[1] pp. 577–591.
 70. A. Pais: *‘Subtle is the Lord...’ The Science and the Life of Albert Einstein* (Clarendon Press, Oxford, 1982).
 71. J.-F. Pascual-Sánchez: *lh theory of hadrons and leptons*. Lett. Nuovo Cim. **28** (1980) 321–324.
 72. J.-F. Pascual-Sánchez: *On the limits of Poincaré gauge theory*. Phys. Lett. **A108** (1985) 387–390.
 73. J.-F. Pascual-Sánchez: *Constrained macroscopic limit of Poincaré gauge theory of gravity*. Lett. Nuovo Cim. **44** (1985) 326–328.
 74. M. Pawłowski and R. Rączka: *A unified conformal model for fundamental interactions without dynamical Higgs field*. Found. Phys. **24** (1994) 1305–1327.
 75. M. Pawłowski and R. Rączka: *A Higgs-free model for fundamental interactions. Part 1: Formulation of the model*. In: ‘Modern Group Theoretical Methods in Physics’, J. Bertrand et al. (eds.) (Kluwer Academic, Dordrecht, 1995) pp. 221–232.
 76. V. Perlick: *Observer fields in Weylian spacetime models*. Class. Quantum

- Grav. **8** (1991) 1369–1385.
77. E.A. Poberii: *Metric-affine scale-covariant gravity*. Gen. Relat. Grav. **26** (1994) 1011–1054.
78. V.N. Ponomariov and Ju. Obukhov: *The generalized Einstein-Maxwell theory of gravitation*. Gen. Relat. Grav. **14** (1982) 309–330.
79. E. Prugovečki: *Principles of Quantum General Relativity* (World Scientific, Singapore, 1995).
80. R. Puntigam, E. Schrüfer, and F.W. Hehl: *The use of computer algebra in Maxwell's theory*. In: 'Computer Algebra in Science and Engineering', J. Fleischer et al. (eds.) (World Scientific, Singapore, 1995) pp. 195–211.
81. H. Rumpf: *On the translational part of the Lagrangian in the Poincaré gauge theory of gravitation*. Z. Naturf. **33a** (1978) 1224–1225.
82. V. de Sabbata and C. Sivaram: *Spin and Torsion in Gravitation* (World Scientific, Singapore, 1994).
83. E. Schrödinger: *Space-Time Structure*. Reprinted with corrections (Cambridge University Press, Cambridge, 1960).
84. E. Schrüfer, F.W. Hehl, and J.D. McCrea: *Exterior calculus on the computer: The REDUCE-package EXCALC applied to general relativity and the Poincaré gauge theory*. Gen. Rel. Grav. **19** (1987) 197–218.
85. E. Sezgin and P. van Nieuwenhuizen: *New ghost-free Lagrangians with propagating torsion*. Phys. Rev. **D21** (1980) 3269–3280.
86. P. Teyssandier and R.W. Tucker: *Gravity, gauges and clocks*. Class. Quantum Grav., to appear (1995/96).
87. W. Thirring: *Classical Field Theory*. A Course in Mathematical Physics 2, 2nd ed. (Springer, New York, 1986).
88. A. Trautman: *On the structure of the Einstein-Cartan equations*. In: 'Differential Geometry', Symposia Mathematica Vol. **12** (Academic Press, London 1973) p. 139–162.
89. R. Tresguerres: *An exact solution of $(2 + 1)$ -dimensional topological gravity in metric-affine spacetime*. Phys. Lett. **A168** (1992) 174–178.
90. R. Tresguerres: *Topological gravity in a 3-dimensional metric-affine space*. J. Math. Phys. **33** (1992) 4231–4238.
91. R. Tresguerres: *Exact vacuum solutions of 4-dimensional metric-affine gauge theories of gravitation*. Z. Phys. **C65** (1995) 347–354.
92. R. Tresguerres, *Exact static vacuum solution of four-dimensional metric-affine gravity with nontrivial torsion*. Phys. Lett. **A 200** (1995) 405–410.
93. R. Tresguerres: *Weyl-Cartan model for the non-massive cosmological eras with a dilatedly polarized vacuum*. Phys. Lett. **A**, submitted (1994).

94. E. Tsantilis, R.A. Puntigam, and F.W. Hehl: *A quadratic curvature Lagrangian of Pawłowski and Rączka: A finger exercise with MathTensor*. In: ‘Relativity and Scientific Computing – Computer Algebra, Numerics, Visualization –’ F.W. Hehl, R.A. Puntigam, H. Ruder (eds.) (Springer, Berlin, 1996) to be published.
95. R.W. Tucker and C. Wang: *Black holes with Weyl charge and non-Riemannian waves*. *Class. Quantum Grav.* **12** (1995) 2587–2605.
96. C. Wiesendanger: *Poincaré gauge invariance and gravitation in Minkowski spacetime*. Electronic archive Los Alamos, gr-qc/9505049 (1995).
97. V.V. Zhytnikov: *Wavelike exact solutions of $R + R^2 + Q^2$ gravity*. *J. Math. Phys.* **35** (1994) 6001–6017.

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